An asymptotic duality between the Oldroyd-Maxwell and grade-two fluid models

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Abstract

We prove an asymptotic relationship between the grade-two fluid model and a class of models for non-Newtonian fluids suggested by Oldroyd, including the upper-convected and lower-convected Maxwell models. This confirms an earlier observation of Tanner. We provide a new interpretation of the temporal instability of the grade-two fluid model for negative coefficients. Our techniques allow a simple proof of the convergence of the steady grade-two model to the Navier-Stokes model as $\alpha \to 0$ (under suitable conditions) in three dimensions. They also provide a proof of the convergence of the steady Oldroyd models to the Navier-Stokes model as their parameters tend to zero.

1 Models for rheology

There are many models of non-Newtonian fluids (see [20, 24] and references in [9]). There has been significant recent progress on the mathematical theory of such models [9, 14]. We propose here to compare two of these models and show that they have a very close relationship in a certain limit. Some of these models may be viewed as a natural perturbation of other models. For example, the grade-two model is a perturbation of the Navier-Stokes model [5], and the grade-three model [1, 9] extends the grade-two model by adding one additional parameter; when that parameter is zero, the grade-three model reduces formally to the grade-two model, just as the grade-two model formally reduces to Navier-Stokes when the appropriate parameters are set to zero. However, it is not at all obvious whether such
Model | Grade Two | Oldroyd
---|---|---
Base model coefficients | $\alpha_1$ | $\lambda_1$
 | $\alpha_2$ | $\mu_1$
Dual model coefficients | $-\eta\lambda_1$ | $\eta(\lambda_1 + \mu_1)$
 | $\eta(\lambda_1 + \mu_1)$ | $-\frac{\alpha_1}{\eta}$ | $\left(\frac{\alpha_1 + \alpha_2}{\eta}\right)$

Table 1: Asymptotic duality relations between the grade-two ($u_G$) and Oldroyd ($u_O$) model solutions for appropriately matched coefficients. The meaning of asymptotic duality is the following. Let $u_G(a, b)$ denote the solution of the grade-two model with coefficients $\alpha_1 = a$, $\alpha_2 = b$, and let $u_O(l, m)$ denote the solution of the Oldroyd model with coefficients $\lambda_1 = l$, $\mu_1 = m$. Then $u_G(\alpha_1, \alpha_2) \approx u_O\left(-\frac{\alpha_1}{\eta}, \frac{(\alpha_1 + \alpha_2)}{\eta}\right)$ and $u_G(-\eta\lambda_1, \eta(\lambda_1 + \mu_1)) \approx u_O(\lambda_1, \mu_1)$. We will prove that these approximations hold rigorously with an error of order $\lambda_1^2$ (or equivalently $\alpha_2^2$) under the simplifying assumption that $\mu_1 = \alpha_1 + \alpha_2 = 0$.

perturbations are singular or not, nor is it clear in what norms convergence would occur as parameters tend to zero.

Similarly, the Oldroyd-B model may be viewed as a perturbation of the upper-convected Maxwell model in the same way. However, it is less obvious that the grade-two model and, say, the Maxwell model could be viewed as perturbations of each other, although this has been known for some time [31]. Our purpose here is to provide a rigorous analysis of such an observation. We will verify that the grade-two model is asymptotically equivalent to certain Oldroyd models under suitable smoothness conditions. We do not rule out the possibility that the models can differ when there are singularities in the solutions, e.g., due to geometric features of the boundary. In Table 1, we delineate the relationships between the grade-two model and the Oldroyd model.

The Maxwell model, originally proposed in the first equation on page 52 of [22], introduces only one additional parameter $\lambda$ (corresponding to $\lambda_1$ in the Oldroyd models) beyond those required to describe a Newtonian fluid. The sign of this parameter has a definite physical significance [19]. The lower-convected and upper-convected Maxwell models may be viewed as simplified versions of the Oldroyd models A and B, respectively, for non-Newtonian fluids [23].

The grade-two model [13] introduces two additional parameters $\alpha_1$ and $\alpha_2$ beyond those of a Newtonian fluid. However, $\alpha_1$ plays a dominant role physically and mathematically. The sign of $\alpha_1$ also has a definite physical significance, as does the parameter $\lambda_1$ in the convected Maxwell models. Indeed, as indicated in Table 1, in the correspondence proved here between the Oldroyd-Maxwell model and the grade-two model, the signs of $\alpha_1$ and $\lambda_1$ are opposite ($\alpha_1 \iff -\eta\lambda_1$). More precisely, we will prove that the grade-two ($u_G$) and Oldroyd ($u_O$) model solutions satisfy (see the explanation of notation in Table 1)

\[ u_G(\alpha_1, -\alpha_1) \approx u_O(-\frac{\alpha_1}{\eta}, 0) + O\left(\frac{\alpha_1^2}{\eta}\right), \]

under certain restrictions.

The first results on existence of solutions for visco-elastic fluid models were presented by Renardy [28, 29]. The first of these articles [28] addresses the upper-convected Maxwell model which has been extensively studied, see e.g. [33, 14, 6].

Reference [25] established the “structural stability” of the upper-convected Maxwell
model, meaning that the solutions depend continuously on the modeling parameter \( \lambda \). Our work can be viewed as an extension of the concept of structural stability to compare not just one model with two different values of the model parameter, but two different models (with appropriately linked parameters), cf. \([4]\).

In \([4]\), it was demonstrated that two simplified models of surface water waves were equally accurate approximations of the physics being modeled over the time scale for which the models are viable. One of the models (KdV) has unusual algebraic properties that allow the study of certain exact solutions analytically, whereas the other model is inherently easier to approximate via numerical simulation. The gist of \([4]\) is that either model can be chosen to fit the need, whether for exact analytical results with simplified initial data or for computational results for arbitrary initial data. In the models studied here, there is a similarity in that we show that certain models are equally accurate at least for small values of the model parameter. However, the divergence of the models considered here is more extreme, in that each of the models exhibits a time-dependent instability that renders it ill-posed depending on the sign of the primary modeling parameter.

It is well known \([10, 12]\) that the grade-two model exhibits a temporal, high-frequency instability for negative \( \alpha_1 \). Unfortunately, this sign is the relevant one for many fluids of interest. This makes the time-dependent model inappropriate in certain contexts. However, we will show that in just this case, the time-independent solutions of the grade-two model are essentially the same as those of the upper-convected Maxwell model, for which there is no time-dependent instability for the corresponding sign of the primary coefficient \( \lambda_1 \). Conversely, when the model parameter has the opposite sign, the upper-convected Maxwell model has a time-dependent instability, whereas the asymptotically equivalent grade-two model does not. Thus one can choose the appropriate model for the appropriate parameter regime as needed.

Physical measurements \([21, 26]\) of rheological properties of fluids typically are done with steady flows. Thus these experiments do not directly impinge upon the question of temporal instabilities of the models. A comparison of experimental results with the grade-two fluid model was done in \([34]\). The grade-two model has been recently used as a model of turbulence \([8, 32, 27]\) as anticipated by Rivlin \([30]\) who observed that “some analogy may exist between the turbulent Newtonian flow and the laminar flow of a non-Newtonian fluid.”

In our proofs, we will limit discussions to domains \( D \) whose boundaries \( \partial D \) are either smooth or convex polyhedra with further restrictions on their inner angles. However, there is no such restriction on the formal arguments. An open question is whether or not singularities in solutions induced by irregular boundaries could modify the conclusions of this article.

### 1.1 Model equations

In most models of fluids, the equations of fluid motion take the form

\[
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nabla \cdot \mathbf{T} + \mathbf{f},
\]

where \( \mathbf{T} \) is called the extra (or deviatoric) stress and \( \mathbf{f} \) is externally given data. In the case of the grade-two model, a time derivative of \( \mathbf{u} \) appears in the expression for \( \mathbf{T} \) as well as explicitly in the convection term. Since the time-dependent models that we consider exhibit an instability for certain parameters, we focus on the time-independent versions of
the models, which take the form

\[ u \cdot \nabla u + \nabla p = \nabla \cdot \mathbf{T} + f, \quad (1.1) \]

and are known to be well-posed in many cases of interest. The models differ only according to the way the stress \( \mathbf{T} \) depends on the velocity \( u \).

For incompressible fluids, the equation (1.1) is accompanied by the condition

\[ \nabla \cdot u = 0, \quad (1.2) \]

which we will assume holds in all of the following discussion. In addition, we will assume that \( u = 0 \) on the boundary \( \partial \mathcal{D} \) of the domain \( \mathcal{D} \) in which (1.1) and (1.2) hold. Our results can be extended to the case where \( u = g \) on \( \partial \mathcal{D} \) provided \( g \cdot n = 0 \). For suitable expressions for \( \mathbf{T} \) defined in terms of \( u \), the problem (1.1) and (1.2) can be shown to be well-posed, as we indicate in special cases.

### 1.2 Notation

We will assume that \( \mathcal{D} \) is a domain in \( \mathbb{R}^d \) for \( d = 2 \) or \( 3 \), with boundary \( \partial \mathcal{D} \) about which restrictions are made in section 1.3. We will use standard Sobolev spaces on \( \mathcal{D} \) with norms

\[ \| \mathbf{T} \|_{W^m_q(\mathcal{D})} = \left( \sum_i \sum_{|\alpha| \leq m} \int_{\mathcal{D}} |D^\alpha T_i(x)|^q \, dx \right)^{1/q}, \quad (1.3) \]

where the indices \( i \) in (1.3) range over the index set of the tensor \( \mathbf{T} \). We also denote by \( L^q(\mathcal{D}) \) the case when \( m = 0 \). We also use the seminorms

\[ |\mathbf{T}|_{W^m_q(\mathcal{D})} = \left( \sum_i \sum_{|\alpha| = m} \int_{\mathcal{D}} |D^\alpha T_i(x)|^q \, dx \right)^{1/q}. \]

When \( q = 2 \), we replace \( W^m_q \) by \( H^m \). Note that for a tensor \( \mathbf{T} \) of arity 2,

\[ \| \nabla \cdot \mathbf{T} \|_{L^q(\mathcal{D})} = \left( \sum_{i=1}^{d} \left( \sum_{j=1}^{d} T_{i,j}(x) \right)^q \, dx \right)^{1/q} \leq \left( d \int_{\mathcal{D}} \sum_{i,j,k=1}^{d} |T_{i,j,k}(x)|^q \, dx \right)^{1/q} \]

\[ \leq d^{1/q} \| \mathbf{T} \|_{W^1_q(\mathcal{D})}. \quad (1.4) \]

Note that \( d^{1/q} \leq d \) for all \( 1 \leq q \leq \infty \).

We define \( H(\text{curl}, \mathcal{D}) \) to be the space

\[ H(\text{curl}, \mathcal{D}) = \{ \mathbf{v} \in L^2(\mathcal{D})^d : \text{curl} \, \mathbf{v} \in L^2(\mathcal{D})^d \} \]

with norm

\[ \| \mathbf{v} \|_{H(\text{curl}, \mathcal{D})} = \sqrt{\| \mathbf{v} \|_{L^2(\mathcal{D})}^2 + \| \text{curl} \, \mathbf{v} \|_{L^2(\mathcal{D})}^2}. \]
1.3 Regularity assumptions on $D$

Reflecting [16], we make the following assumptions. Consider the elliptic equations

$$v - \Delta v = f \quad \text{in } D$$
$$\nabla v \cdot n = 0 \quad \text{on } \partial D,$$

(1.5)

where $n$ is the unit outer normal to $\partial D$, and

$$-\Delta v = f \quad \text{in } D$$
$$v = 0 \quad \text{on } \partial D.$$

(1.6)

We introduce the following condition: suppose that the domain $D$ has the property that there is a constant $C$ such that each problem (1.5) and (1.6) has a unique solution $v \in H^2(D)$ for all $f \in L^2(D)$ satisfying

$$\|v\|_{H^2(D)} \leq C \|f\|_{L^2(D)}.$$

(1.7)

Similarly, we consider a Stokes system,

$$-\Delta v + \nabla p = f \quad \text{in } D$$
$$\nabla \cdot v = 0 \quad \text{in } D,$$

(1.8)

$$v = 0 \quad \text{on } \partial D.$$

Let $q_0 = 1$ in two dimensions ($d = 2$) and $q_0 = 6/5$ in three dimensions ($d = 3$). For all $q > q_0$ (and $q \geq q_0$ if $d = 3$), if $f \in L^q(D)$, then $f \in H^{-1}(D)$. So we know in this case that (1.8) is well-posed in $H^1(D)^d \times L^2(D)/\mathbb{R}$ for all $f \in L^q(D)^d$. We introduce the following condition: suppose that, for some $q > q_0$ ($q \geq q_0$ if $d = 3$), the domain $D$ has the property that there is a constant $C_q$ such that for all $f \in L^q(D)^d$ there is a unique pair $v \in W^{2,q}_d(D)^d$ and $p \in W^{1,q}_d(D)/\mathbb{R}$ solving (1.8) and satisfying

$$\|v\|_{W^{2,q}_d(D)} + \|p\|_{W^{1,q}_d(D)/\mathbb{R}} \leq C_{q,D} \|f\|_{L^q(D)} \quad \text{for all } f \in L^q(D)^d.$$

(1.9)

There are many sufficient conditions known that guarantee (1.7) or (1.9) [7, 11]. Ultimately, in order to use Lemma 2.1 below, we must also assume further restrictions.

Finally, we will assume that

$$\|v\|_{H^3(D)} + \|p\|_{H^1(D)/\mathbb{R}} \leq C_3 \|f\|_{H^1(D)},$$

(1.10)

which requires additional smoothness on $\partial D$, for one key result.

1.4 Sobolev inequalities

We will use the Sobolev inequality

$$\|T\|_{W^{3,q}_d(D)} \leq c_q \|T\|_{W^{2,q+1}_d(D)}$$

(1.11)

for tensor functions $T$ of arity less than 3, in dimension $d \leq 3$, and for $0 \leq s \leq 1$, with $q > d$. 

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We recall from [17] the inequality
\[ \| uv \|_{L^2(D)} \leq \sigma_q \| u \|_{L^q(D)} \| v \|_{H^1(D)}, \] (1.12)
valid provided \( q > d \) for \( d = 2 \) or \( q \geq d \) for \( d \geq 3 \), where \( \sigma_q \) is a Sobolev constant. An immediate consequence is
\[ \left| \int_D u(x)v(x)w(x) \, dx \right| \leq \sigma_q \| u \|_{L^q(D)} \| v \|_{H^1(D)} \| w \|_{L^2(D)}. \] (1.13)

In addition, we scale the domain so that we can assume that \( D = \int_D dx \leq 1 \). This allows the removal of one constant in the forthcoming estimates, since in this case
\[ \| u \|_{L^2(D)} \leq \| u \|_{L^q(D)} \] (1.14)
for \( q > 2 \).

2 Grade-two model

The grade-two fluid model [13] has the following constitutive equation for the extra stress tensor \( T_G \):
\[ T_G = \eta A_1 + \alpha_1 \frac{\Delta}{\Delta t} A_1 + \alpha_2 A_1^2, \]
where \( L = L(u) = \nabla u, A_1 = A_1(u) = L^t + L = (\nabla u)^t + \nabla u \), and the material derivative and the lower-convected Oldroydian derivative are given by
\[ \frac{D}{Dt} f := \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) f, \quad \frac{\Delta}{\Delta t} f := \frac{D}{Dt} f + f(\nabla u) + (\nabla u)^t f, \]
for any tensor-valued function \( f \). In the time-independent case, this simplifies to
\[ T_G = \eta A_1 + \alpha_1 (u \cdot \nabla A_1 + A_1L + L^t A_1) + \alpha_2 A_1^2 \]
\[ = \eta A_1 + \alpha_1 (u \cdot \nabla A_1 + A_1L + L^t A_1 - A_1^2) + (\alpha_1 + \alpha_2) A_1^2. \]
Expanding, we find
\[ A_1L + L^t A_1 - A_1^2 = (L + L^t)L + L^t(L + L^t) - (L + L^t)(L + L^t) \]
\[ = L'L - LL'. \]
Define \( R = R(u) = \frac{1}{2} (\nabla u' - \nabla u) \). Then
\[ 2 (RA_1 + A_1R^t) = (L^t - L)(L + L^t) + (L + L^t)(L - L^t) \]
\[ = L'L - LL' - LL' + L'L = 2 (L'L - LL'). \] (2.1)

Therefore
\[ A_1L + L^t A_1 - A_1^2 = RA_1 + A_1R^t, \]
and we thus find (in the time-independent case)
\[ T_{\alpha} = \eta A_1 + \alpha_1 (u \cdot \nabla A_1 + RA_1 + A_1 R^t) + (\alpha_1 + \alpha_2) A_1^2 \]
\[ = \eta((\nabla u)^t + \nabla u) + \alpha_1 G(u, A_1(u), 1 + \alpha_2/\alpha_1), \]  
(2.2)
where we define \( G : W^2_q(D)^d \times W^1_q(D)^{d^2} \times \mathbb{R} \to L^q(D) \) (for \( q > d \) via
\[ G(v, U, \tau) = v \cdot \nabla U + R(v)U + UR(v)^t + \frac{1}{2} \tau (A_1(v)U + UA_1(v)), \]
(2.3)
where we recall that \( A_1(v) = (\nabla v)^t + \nabla v \) and \( R(v) = \frac{1}{2}((\nabla v)^t - \nabla v) \). Note that \( G \) is bilinear in \( v \) and \( U \), and in particular
\[ G(v, aU^1 + U^2, \tau) = aG(v, U^1, \tau) + G(v, U^2, \tau), \]
(2.4)
for any \( v \in W^2_q(D)^d, U^1, U^2 \in W^1_q(D)^{d^2} \) and \( a \in \mathbb{R} \). In addition,
\[ \| G(v, U, \tau) \|_{L^q(D)} \leq C(1 + \tau) \| v \|_{W^2_q(D)} \| U \|_{W^1_q(D)}. \]
(2.5)

The following collects results proved in [15, 3, 9].

**Lemma 2.1** Suppose that \( \alpha_2 = -\alpha_1 \) and that \( D \) is either smooth or a convex polyhedron. For \( d = 3 \), we further assume that all inner angles are less than \( 3\pi/4 \). Then there are constants \( \eta_0 > 0, \alpha_0 > 0, q_0 > 2, \) and \( C > 0 \) such that, for any \( f \in H(\text{curl}, D) \) satisfying
\[ \| f \|_{H(\text{curl}, D)} \leq C, \]  
the grade-two system
\[ -\eta \Delta u + u \cdot \nabla u + \nabla p - \alpha_1 \nabla \cdot G(u, A_1(u), 0) = f \text{ in } D \]
\[ \nabla \cdot u = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D \]  
(2.6)
has a unique solution with
\[ u \in \begin{cases} 
W^2_q(D)^2 & \text{for } q_0 > q > 2 & d = 2 \text{ (see [15])} \\
W^1_{\infty}(D)^3 \cap W^2_q(D)^3 & d = 3 \text{ (see [3])}
\end{cases} \]
and \( p \in H^1(D)/\mathbb{R} \) for all \( 0 < |\alpha_1| < \alpha_0 \) and \( \eta \geq \eta_0 \).

We emphasize that we cannot assert that \( \| u \|_{W^2_q(D)} \) remains bounded as \( \alpha_1 \to 0 \), but fortunately this is not needed in the subsequent arguments. Indeed, it would be unlikely that this would hold under the assumptions given, since we do not assume that \( f \in L^q(D)^d \) for any \( q > 2 \). Only the special structure of the grade-two model provides the extra regularity.

### 3 Oldroyd’s models

An important five parameter subset of the eight parameter model of Oldroyd [23] has a constitutive relation of the form
\[ T + \lambda_1 (u \cdot \nabla T + RT + TR^t) - \mu_1 (ET + TE) \\
= 2\eta (E + \lambda_2 (u \cdot \nabla E + RE + ER^t) - 2\mu_2 E^2), \]  
(3.1)
where the three parameters \( \mu_0, \nu_0, \) and \( \nu_1 \) in [23] are set to zero and we define \( E = E(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^t) = \frac{1}{2}A_1 \) (recall that \( R = R(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u}^t - \nabla \mathbf{u}) \)). Note that \( E^t = E, R + E = \nabla \mathbf{u}^t, \) and \( R - E = -\nabla \mathbf{u} \).

We can relate the Maxwell models to the Oldroyd scheme simply. The upper-convected model is the case \( \lambda_1 = \mu_1 \) and the lower-convected model is the case \( \lambda_1 = -\mu_1, \) both with \( \lambda_2 = \mu_2 = 0. \) Thus from now on, we make the assumption that \( \lambda_2 = \mu_2 = 0. \)

In the case \( \lambda_2 = \mu_2 = 0 \), (3.1) can be written
\[
T + \lambda_1(\mathbf{u} \cdot \nabla T + RT + TR^t) - \mu_1(ET + TE) = T + \lambda_1G(\mathbf{u}, T, -\mu_1/\lambda_1) = 2\eta E, \tag{3.2}
\]
where \( G \) is defined in (2.3).

### 3.1 Bounds for Oldroyd

The following is proved in [16, Theorem 5.8] and [17, Theorem 5.8].

**Lemma 3.1** Suppose that \( q > d \) and the conditions (1.7) and (1.9) hold. There are constants \( \eta_0 > 0, \mu_0 > 0, \lambda_0 > 0, C_1 > 0 \) and \( C_2 > 0 \) such that, for any \( f \in W^1_q(D)^d \) satisfying \( \|f\|_{W^1_q(D)} \leq C_2, \) the Oldroyd system (1.1), (3.2) has a solution for all \( |\lambda_1| < \lambda_0\eta, |\mu_1| \leq \mu_0|\lambda_1|, \) and \( \eta \geq \eta_0, \) satisfying
\[
\eta(\|\mathbf{u}\|_{W^1_q(D)} + \|T\|_{W^1_q(D)}) + \|p\|_{W^1_q(D)/\mathbb{R}} \leq C_1\|f\|_{W^1_q(D)}, \tag{3.3}
\]
where \( C_1 \) is independent of \( \lambda_1 \) and \( \mu_1. \)

A simple consequence of (3.3) is that
\[
\eta\|\mathbf{u}\|_{W^1_q(D)} \leq C_1C_2.
\]

It appears that the relationship between \( f \) and \( \mathbf{u} \) is suboptimal in terms of regularity. This is a consequence of our approach in [16, Lemma 4.1] and [17, Lemma 4.1] to deal with the lack of explicit dissipation in certain Oldroyd models.

A key step in the proof of Lemma 3.1 is a wellposedness result for the transport equation with an arbitrary advection velocity \( \mathbf{v}, \) which follows from the techniques in [2, 18] and is detailed in [16, Lemma 4.1] and [17, Lemma 4.1]. The following is an example of such a result and will be proved in section 7.2.

**Lemma 3.2** Suppose that \( 2 \leq d \leq 4, q \geq 2, \) and \( D \) is bounded and Lipschitz. Suppose further that \( \mathbf{u} \in W^{1}_{\infty}(D)^d, \) with \( \nabla \cdot \mathbf{u} = 0 \) in \( D, \) \( \mathbf{u} \cdot \mathbf{n} = 0 \) on \( \partial D, \) and
\[
\|\nabla \mathbf{u}\|_{L^{\infty}(D)} \leq \frac{1}{4 + 2(|\lambda_1| + |\mu_1|)}, \quad \|\mathbf{u}\|_{W^2_q(D)} \leq \frac{1}{(4\sigma_q + c_q)2(|\lambda_1| + |\mu_1|)}, \tag{3.4}
\]
where the Sobolev constants \( c_q \) and \( \sigma_q \) are defined in section 1.4. Then for each \( \mathbf{G} \in L^2(D)^d, \) there is a unique solution \( T \in L^2(D)^d \) of the equation
\[
T + \lambda_1(\mathbf{u} \cdot \nabla T + R(\mathbf{u})T + TR(\mathbf{u})^t) - \mu_1(TE(\mathbf{u}) + E(\mathbf{u})T) = \mathbf{G}, \tag{3.5}
\]
where we recall that \( R(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u}^t - \nabla \mathbf{u}) \) and \( E(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u}^t + \nabla \mathbf{u}^t), \) satisfying \( T \in H^1(D)^d \) when \( \mathbf{G} \in H^1(D)^d \) with the bounds
\[
\|T\|_{L^q(D)} \leq \left(1 + \frac{1}{2}(|\lambda_1| + |\mu_1|)\right)\|\mathbf{G}\|_{L^q(D)},
\]
\[
\|T\|_{H^1(D)} \leq \left(2 + |\lambda_1| + |\mu_1|\right)\|\mathbf{G}\|_{H^1(D)}. \tag{3.6}
\]
Lemma 3.3 Suppose that $2 \leq d \leq 4$, $q \geq 2$, $\tilde{\mu} \in \mathbb{R}$, and $\mathcal{D}$ is bounded and Lipschitz. Suppose further that $v \in W_\infty^1(\mathcal{D})^d$, with $\nabla \cdot v = 0$ in $\mathcal{D}$, $v \cdot n = 0$ on $\partial \mathcal{D}$, and

$$\| \nabla v \|_{L^\infty(\mathcal{D})} \leq \frac{1 - c_0}{|1 + \tilde{\mu}| + |1 - \tilde{\mu}|}, \text{ where } 0 < c_0 < 1.$$ 

Then for each $G \in L^q(\mathcal{D})^{d^2}$, there is a unique solution $T \in L^q(\mathcal{D})^{d^2}$ of the equation

$$T + v \cdot \nabla T + R(v)T + TR(v)^t - \tilde{\mu}(TE(v) + E(v)T) = G,$$  \hspace{1cm} (3.7)

satisfying

$$\max \{\| T \|_{L^q(\mathcal{D})}, \frac{1}{3} \| v \cdot \nabla T \|_{L^q(\mathcal{D})}\} \leq \frac{1}{c_0} \| G \|_{L^q(\mathcal{D})}.$$  \hspace{1cm} (3.8)

This will be applied with $\tilde{\mu} = \mu_1 / \lambda_1$.

### 3.2 Oldroyd as a perturbation of Navier-Stokes

We can characterize the Oldroyd problem as a perturbation of the Navier-Stokes equations via

$$-\eta \Delta u_o + u_o \cdot \nabla u_o + \nabla p_o = \lambda_1 \nabla \cdot M_o + f$$

$$\nabla \cdot u_o = 0,$$  \hspace{1cm} (3.9)

where we define the tensor

$$M_o = \frac{1}{\lambda_1} (T_o - 2\eta E(u_o))$$  \hspace{1cm} (3.10)

and we denote by $T_o$ the solution of (3.2) with $u = u_o$. Using (2.3) and (2.4), we see that $M_o$ satisfies

$$M_o + \lambda_1 (u_o \cdot \nabla M_o + R(u_o)M_o + M_oR(u_o)^t) - \mu_1 (E(u_o)M_o + M_oE(u_o))$$

$$= M_o + \lambda_1 G(u_o, M_o, -\mu_1 / \lambda_1)$$

$$= \frac{1}{\lambda_1} (T_o - 2\eta E(u_o)) + G(u_o, T_o, -\mu_1 / \lambda_1) - G(u_o, 2\eta E(u_o), -\mu_1 / \lambda_1).$$  \hspace{1cm} (3.11)

Applying (3.2), (2.4) and (2.3) to (3.11), we find

$$M_o + \lambda_1 (u_o \cdot \nabla M_o + R(u_o)M_o + M_oR(u_o)^t) - \mu_1 (E(u_o)M_o + M_oE(u_o))$$

$$= -2\eta G(u_o, E(u_o), -\mu_1 / \lambda_1)$$

$$= -2\eta (u_o \cdot \nabla E(u_o) + R(u_o)E(u_o) + E(u_o)R(u_o)^t - (\mu_1 / \lambda_1)E(u_o)^2).$$  \hspace{1cm} (3.12)

Setting $v = \lambda_1 u_o$, we conclude from Lemma 3.3, Lemma 3.2, and Lemma 3.1, if their assumptions hold, that

$$\max \{\| M_o \|_{L^2(\mathcal{D})}, \frac{1}{3} \| \lambda_1 \| u_o \cdot \nabla M_o \|_{L^2(\mathcal{D})}\}$$

$$\leq \frac{2\eta}{c_0} \| u_o \cdot \nabla E(u_o) + R(u_o)E(u_o) + E(u_o)R(u_o)^t - (\mu_1 / \lambda_1)E(u_o)^2 \|_{L^2(\mathcal{D})}$$

$$\leq C \| f \|_{W_\infty^4(\mathcal{D})}^2,$$  \hspace{1cm} (3.13)
for some \( q > d \), at least for \( \lambda_1 \) and \( f \) sufficiently small and \( \mu_1 \) bounded as in Lemma 3.1.

Let \( u_N \) denote the solution to the Navier-Stokes equations
\[
-\eta \Delta u_N + u_N \cdot \nabla u_N + \nabla p_N = f
\]
subject to Dirichlet boundary conditions \( u_N = 0 \) on \( \partial D \).

Let \( \sigma \) be the smallest constant such that
\[
| (w \cdot \nabla v, w) | \leq \sigma |v|_{H^1(D)} |w|_{H^1(D)}^2 \quad \forall \, v, w \in H^1_0(D)^d.
\]

The following theorem is a consequence of (3.9) and (3.13).

**Theorem 3.4** Suppose that the assumptions of Lemma 3.1 hold. Let \( u_o \) be a solution of (1.1) and (3.2) guaranteed by Lemma 3.1. For any solution \( u_N \) of (3.14) such that
\[
|u_N|_{H^1(D)} \leq \eta/2\sigma,
\]
where \( \sigma \) is defined in (3.15), we have
\[
\eta |u_o - u_N|_{H^1(D)} \leq 2C|\lambda_1| \| f \|_{W^1_q(D)}^2,
\]
where the constant \( C \) is from (3.13).

The convergence of the grade-two model to Navier-Stokes was known [15, 5] as \( \alpha \to 0 \). But [16] and [17] are the first complete proofs regarding wellposedness of the 3-parameter Oldroyd system, which lack explicit dissipation, so this is the first proof of convergence for this Oldroyd model to Navier-Stokes that we are aware of. Estimates in the full \( H^1(D) \) norm follow from (3.16) and (3.17) from Poincaré’s inequality.

**Proof.** Let \( w = u_o - u_N \). From (3.9) and (3.14), we have
\[
-\eta \Delta w + u_o \cdot \nabla u_o - u_N \cdot \nabla u_N + \nabla (p_o - p_N) = \lambda_1 \nabla \cdot M_o,
\]
as well as \( \nabla \cdot w = 0 \). Using the definition of \( \sigma \) in (3.15), we get
\[
| (u_o \cdot \nabla u_o - u_N \cdot \nabla u_N, w) | = | (w \cdot \nabla u_N, w) | \leq \sigma |u_N|_{H^1(D)} |w|_{H^1(D)}^2.
\]

Using assumption (3.16), we have
\[
| (u_o \cdot \nabla u_o - u_N \cdot \nabla u_N, w) | \leq (\eta/2) |w|_{H^1(D)}^2.
\]

Thus standard estimates for Navier-Stokes equation imply
\[
\frac{\eta}{2} |w|_{H^1(D)} \leq |\lambda_1| \| \nabla \cdot M_o \|_{H^{-1}(D)} \leq |\lambda_1| \| M_o \|_{L^2(D)} \leq C|\lambda_1| \| f \|_{W^1_q(D)}^2,
\]
in view of (3.13). QED

**Remark 3.5** The proof of Theorem 3.4 also shows that, if there is a solution \( u_N \) for the Navier-Stokes fluid model satisfying the condition (3.16), then it must be unique.
4 Grade two/Oldroyd comparison

Now let us relate the grade-two models in the case of general $\alpha_1$ and $\alpha_2$ to the Oldroyd models. From (2.2), we see that the extra-stress tensor $T_G$ reads

$$T_G = 2\eta E(u_G) + \alpha_1 G(u_G, A_1(u_G), 1 + \alpha_2/\alpha_1).$$

This should be compared with the corresponding time-independent Oldroyd model (3.2):

$$T_O = 2\eta E(u_O) - \lambda_1 G(u_O, T_O, -\mu_1/\lambda_1).$$

(4.1)

For $\lambda_1$ and $\mu_1$ small, we expect the solution to (3.2) to be asymptotically

$$T_O \approx 2\eta E(u_O) = \eta A_1(u_O),$$

so that the next order of approximation, based on (4.1) is

$$T_O \approx 2\eta E(u_O) - \lambda_1 \eta G(u_O, A_1(u_O), -\mu_1/\lambda_1).$$

(4.2)

Thus the general grade-two model corresponds to the Oldroyd model with $\lambda_1 \iff -\alpha_1/\eta$ and $\mu_1 \iff (\alpha_1 + \alpha_2)/\eta$. These relations can be inverted to write

$$\alpha_1 \iff -\eta \lambda_1 \text{ and } \alpha_2 \iff \eta (\lambda_1 + \mu_1).$$

(4.3)

We collect these relations in Table 1.

5 Rigorous verification

We now demonstrate that the Oldroyd models and grade-two models are asymptotically similar for steady solutions, with coefficients related as in Table 1, provided that the grade-two model has a smooth solution and a certain smoothness condition holds for the Oldroyd model. Smooth solutions for grade-two are known in a special case, namely $\alpha_2 = -\alpha_1$, and we write $\alpha$ for $\alpha_1$. (However, we do not assume any particular sign for $\alpha$, although we will eventually choose $|\alpha|$ to be small.) In this case, we have $\mu_1 = 0$ in the Oldroyd model.

5.1 Oldroyd as perturbation of grade-two

For arbitrary $\lambda_1 \in \mathbb{R}$, we take $\alpha = -\eta \lambda_1$, so that $\lambda_1 = -\alpha/\eta$. In this case, (3.12) simplifies to

$$M_o + \lambda_1 (u_o \cdot \nabla M_o + R(u_o)M_o + M_o R(u_o)^t)$$

$$= -2\eta (u_o \cdot \nabla E(u_o) + R(u_o)E(u_o) + E(u_o)R(u_o)^t)$$

$$= -\eta (u_o \cdot \nabla A_1(u_o) + R(u_o)A_1(u_o) + A_1(u_o)R(u_o)^t)$$

$$= -\eta (u_o \cdot \nabla (\nabla u_o + \nabla u_o) + \nabla u_o \nabla u_o - \nabla u_o \nabla u_o),$$

(5.1)
in view of (2.1). Lemmas 3.3 and 3.1 imply that
\[
\begin{align*}
\max \left\{ \| M_o \|_{L^q(D)}, \frac{1}{2} |\lambda_1| \| u_o \cdot \nabla M_o \|_{L^q(D)} \right\} \\
\leq \frac{\eta}{c_0} \| u_o \cdot \nabla (\nabla u_o + \nabla u_o^t) + \nabla u_o^t \nabla u_o - \nabla u_o \nabla u_o^t \|_{L^q(D)} \\
\leq 2 \frac{\eta}{c_0} \left( \| u_o \|_{L^\infty(D)} \| u_o \|_{W^{2}_{2}(D)} + \| \nabla u_o \|_{L^\infty(D)} \| u_o \|_{W^{2}_{2}(D)} \right) \\
\leq 4c_q \frac{\eta}{c_0} \left( \| u_o \|_{W^{2}_{2}(D)} \| u_o \|_{W^{2}_{2}(D)} \right) \leq \frac{4c_q C_1^2}{\eta c_0} \| f \|_{W^{2}_{2}(D)},
\end{align*}
\]
provided that \( \| f \|_{W^{2}_{2}(D)} \leq C_2 \) and
\[
|\lambda_1| \| \nabla u_o \|_{L^q(D)} \leq \frac{1}{2}(1 - c_0),
\]
where \( C_1 \) and \( C_2 \) are the constants in Lemma 3.1, \( c_q \) is the Sobolev constant in (1.11), and \( c_0 \) is the constant in Lemma 3.3. Moreover,
\[
\lambda_1 M_o = -\lambda_1^2 \left( u_o \cdot \nabla M_o + R(u_o) M_o + M_o R(u_o)^t \right) \\
+ \alpha \left( u_o \cdot \nabla A_1(u_o) + R(u_o) A_1(u_o) + A_1(u_o) R(u_o)^t \right).
\]
Substituting (5.3) in (3.9), we find that the Oldroyd model can be written as a perturbed grade-two model:
\[
\begin{align*}
\nabla \cdot u_o - \eta \Delta u_o + \nabla p_o - \alpha \nabla \cdot \left( u_o \cdot \nabla A_1(u_o) + R(u_o) A_1(u_o) + A_1(u_o) R(u_o)^t \right) \\
= f - \lambda_1^2 \nabla \cdot \left( u_o \cdot \nabla M_o + R(u_o) M_o + M_o R(u_o)^t \right).
\end{align*}
\]
On the other hand, according to (1.1) and (2.2), the grade-two model \( u_g \) satisfies
\[
\begin{align*}
\nabla \cdot u_g - \eta \Delta u_g + \nabla p_g - \alpha \nabla \cdot \left( u_g \cdot \nabla A_1(u_g) + R(u_g) A_1(u_g) + A_1(u_g) R(u_g)^t \right) \\
= f.
\end{align*}
\]
Equations (5.4) and (5.5) demonstrate that \( u_o \) satisfies the same differential equation as \( u_g \) but with a different right-hand side in \( H^{-1}(D)^d \). The difference between the two right-hand sides can be bounded in \( H^{-1}(D)^d \) as follows.

### 5.2 Bounds for the perturbation

First, observe that
\[
\begin{align*}
\| \nabla \cdot \left( u_o \cdot \nabla M_o + R(u_o) M_o + M_o R(u_o)^t \right) \|_{H^{-1}(D)} \\
\leq \| u_o \cdot \nabla M_o + R(u_o) M_o + M_o R(u_o)^t \|_{L^2(D)}.
\end{align*}
\]
From (5.2) we have
\[
\| u_o \cdot \nabla M_o \|_{L^q(D)} \leq \frac{12c_q C_1^2}{c_0 \eta |\lambda_1|} \| f \|_{W^{2}_{2}(D)},
\]
and since \( \| R(u) \|_{L^\infty(D)} \leq \| \nabla u \|_{L^\infty(D)} \), we have from (3.3) and (5.2) that
\[
\begin{align*}
\| R(u_o) M_o + M_o R(u_o)^t \|_{L^q(D)} &\leq 2c_q \| u_o \|_{W^{2}_{2}(D)} \| M_o \|_{L^q(D)} \\
&\leq \frac{2c_q C_1}{\eta} \| f \|_{W^{2}_{2}(D)} \| M_o \|_{L^q(D)} \leq \frac{8C_1^3 c_q^2}{c_0 \eta^2} \| f \|_{W^{2}_{2}(D)}^3.
\end{align*}
\]
where $c_q$ is the Sobolev constant in (1.11), under the conditions of Lemmas 3.2 and 3.1. Hence, the difference in right-hand sides in (5.4) and (5.5) is bounded by

$$C \left( \frac{\lambda_1^2}{\eta^2} \| f \|_{W_q^3(D)}^3 + \frac{|\lambda_1|}{\eta} \| f \|_{W_q^3(D)} \right).$$

(5.9)

Here, a power of $|\lambda_1|$ is lost when dividing by $|\lambda_1|$ in (5.7). But on the other hand, we can derive an improved estimate with a higher power of $\lambda_1$ under an appropriate assumption. Note that the equation (5.1) for $M$ is of the form (3.5) with $u = u_o, \mu_1 = 0$, and

$$G = - \eta (u_o \cdot \nabla (\nabla u_o + \nabla u_o^t) + \nabla u_o \nabla u_o^t - \nabla u_o \nabla u_o^t).$$

Thus (3.6) implies that

$$\| \nabla M_o \|_{L^2(D)} \leq 2(1 + 2|\lambda_1|) \| G \|_{H^1(D)} \leq c M \eta \| u_o \|_{W_q^3(D)} \| u_o \|_{H^3(D)},$$

(5.10)

provided that $\| \nabla u_o \|_{L^\infty(D)} \leq (4 + 2|\lambda_1|)^{-1}$. Suppose that we know that

$$\| u_o \|_{H^3(D)} \leq K,$$

(5.11)

independent of $\lambda_1$. Then we can use (5.10) to replace (5.7) by

$$\begin{align*}
\| u_o \cdot \nabla M_o \|_{L^2(D)} &\leq c M \eta \| u_o \|_{L^\infty(D)} \| u_o \|_{W_q^3(D)} \| u_o \|_{H^3(D)} \\
&\leq c M c_q \eta \| u_o \|_{W_q^3(D)} \| u_o \|_{H^3(D)} \\
&\leq \frac{c M c_q C_q^2 K}{\eta} \| f_o \|_{W_q^3(D)},
\end{align*}$$

(5.12)

where we have invoked (5.11), again used the Sobolev inequality (1.11), and used (3.3) at the last step. In section 7, we will give conditions to insure that $\| u_o \|_{H^3(D)} \leq K$ as $\lambda_1 \to 0$. Combining (5.12) and (5.8), we get

$$\begin{align*}
\| u_o \cdot \nabla M_o + R(u_o)M_o + M_oR(u_o)^t \|_{L^2(D)} &\leq \frac{8C_q}{c_0 \eta^2} \| f \|_{W_q^3(D)} + \frac{c M c_q C_q^2 K}{\eta} \| f_o \|_{W_q^3(D)}.
\end{align*}$$

In this case, the difference in right-hand sides in (5.4) and (5.5) is bounded by

$$\begin{align*}
\lambda_1^2 \| u_o \cdot \nabla M_o + R(u_o)M_o + M_oR(u_o)^t \|_{L^2(D)} &\leq \lambda_1^2 C_q^2 \frac{c_q}{\eta} \| f \|_{W_q^3(D)} \left( \frac{8C_q}{c_0 \eta^2} \| f \|_{W_q^3(D)} + c M K \right).
\end{align*}$$

(5.13)

### 5.3 Main results

Let $\kappa$ be the smallest constant such that the Sobolev-type inequality

$$| (w \cdot \nabla v, w) - \alpha \sum_{k=1}^d \left( (\nabla w) v_{,k} - (\nabla v) w_{,k} - (\nabla v_{,k}) w, w_{,k} \right) | \leq \kappa \| v \|_{W_q^2(D)} \| w \|_{H^1(D)}$$

holds for all $v \in W_q^2(D)^d$ and $w \in H^1_0(D)^d$, which is finite in view of Sobolev’s inequality.

The following theorem is a consequence of the Lipschitz continuity (6.4) of the solution operator for the grade-two model from $H^{-1}(D)^d$ to $H^1_0(D)^d$, that will be proved in Theorem 6.1.
**Theorem 5.1** Suppose that the assumptions of Lemmas 2.1 and 3.1 hold. There are constants \( \eta_0 > 0 \) and \( \Lambda > 0 \) such that for \( \eta \geq \eta_0 \) and \( \lambda_1 \in \mathbb{R} \) with \( 0 < |\lambda_1| \leq \Lambda \), if \( \mathbf{u}_o \) solves (1.1) and (3.2) with \( \mu_1 = 0 \), satisfying
\[
\| \mathbf{u}_o \|_{W^2_1(D)} \leq \frac{\eta}{2\kappa}, \tag{5.15}
\]
where \( \kappa \) is defined in (5.14), and \( \mathbf{u}_G \in H^2(D)^d \) is a solution of the grade-two problem (2.6) with \( \alpha_1 = -\eta \lambda_1 \) and \( \alpha_2 = \eta \lambda_1 \), then
\[
\eta\| \mathbf{u}_o - \mathbf{u}_G \|_{H^1(D)} \leq 2C \left( \lambda_1^2 \| \mathbf{f} \|_{W^1_2(D)}^2 + |\lambda_1| \| \mathbf{f} \|_{W^3_2(D)}^2 \right), \tag{5.16}
\]
where \( C \) is the constant in (5.9). If in addition \( \| \mathbf{u}_o \|_{H^2(D)} \leq K \) holds as \( \lambda_1 \to 0 \), then
\[
\eta\| \mathbf{u}_o - \mathbf{u}_G \|_{H^1(D)} \leq 2C' \lambda_1^2 \| \mathbf{f} \|_{W^1_2(D)} \left( K + \| \mathbf{f} \|_{W^3_2(D)} \right), \tag{5.17}
\]
where \( C' \) is an upper-bound for the constants in (5.9) and (5.12).

In Theorem 3.4, we show that \( \mathbf{u}_o \) tends to the solution of the Navier-Stokes equation \( \mathbf{u}_N \) as \( \lambda_1 \to 0 \). In two dimensions, it was also known [15] that \( \mathbf{u}_G \) tends to the solution of the Navier-Stokes equation \( \mathbf{u}_N \) as \( \alpha \to 0 \). In the time dependent case, this is also proved in [5] in three dimensions. In view of Theorem 3.4 and Theorem 5.1, we obtain an independent verification that this holds as well in three dimensions in the time independent case. We state this result, obtained by combining (3.17) and (5.16), as the following.

**Theorem 5.2** Suppose that the assumptions of Lemmas 2.1 and 3.1 hold. Let \( \mathbf{u}_G \) be a solution of (2.6) guaranteed by Lemma 2.1. For any solution \( \mathbf{u}_N \) of (3.14) such that (3.16) holds, where \( \sigma \) is defined in (3.15), we have
\[
\eta\| \mathbf{u}_G - \mathbf{u}_N \|_{H^1(D)} \leq 2C \left( 2|\lambda_1| \| \mathbf{f} \|_{W^1_2(D)} + \lambda_1^2 \| \mathbf{f} \|_{W^3_2(D)}^2 \right), \tag{5.18}
\]
where the constant \( C \) is the maximum of the constants from (3.13) and (5.16). Note that \( \lambda_1 = -\alpha = -\alpha_1 \) here.

Suppose there exist functions \( \mathbf{u}'_o \) and \( \mathbf{u}'_G \) in \( H^1(D)^d \), independent of \( \lambda_1 \), such that
\[
\mathbf{u}_X = \mathbf{u}_N + \lambda_1 \mathbf{u}'_X + o(\lambda_1), \tag{5.19}
\]
where \( X \) stands for either \( O \) or \( G \). Then, provided \( \| \mathbf{u}_o \|_{H^3(D)} \leq K \) holds as \( \lambda_1 \to 0 \), Theorem 5.1 implies that \( \mathbf{u}'_G = \mathbf{u}'_G' \):
\[
\mathbf{u}'_G - \mathbf{u}'_G = \frac{1}{\lambda_1} \left( (\mathbf{u}_G - \mathbf{u}_N + o(\lambda_1)) - (\mathbf{u}_o - \mathbf{u}_N + o(\lambda_1)) \right) = \frac{1}{\lambda_1} (\mathbf{u}_G - \mathbf{u}_o) + o(1) \to 0
\]
as \( \lambda_1 \to 0 \). This is depicted in Figure 1.
6 Lipschitz estimates for grade-two

To prove inequalities (5.16) and (5.17), we demonstrate a Lipschitz continuity estimate of the form (6.4) below. Define the grade-two fluid operator

\[ \mathcal{G}(v, q) = -\eta \Delta v + v \cdot \nabla v + \nabla q - \alpha \nabla \cdot (v \cdot \nabla A_1(v)) + R(v)A_1(v) + A_1(v)R(v)^t, \]  

(6.1)

where \( R(v) = \frac{1}{2}(\nabla v^t - \nabla v) \) and \( A_1(v) = \nabla v^t + \nabla v \). A natural domain for \( \mathcal{G} \) is the space

\[ \mathcal{V} = \{ (v, q) : v \in H^2(\mathcal{D}), q \in H^1(\mathcal{D})/\mathbb{R}, \nabla \cdot v = 0 \text{ in } \mathcal{D} \}, \]  

(6.2)

and \( \mathcal{G} \) maps \( \mathcal{V} \) to \( H^{-1}(\mathcal{D}) \). Thus we consider two pairs of functions \((u^i, p^i) \in \mathcal{V}\) and define \( f^i = \mathcal{G}(u^i, p^i) \). In one application, \( u^1 = u_0 \) and \( u^2 = u_G \).

**Theorem 6.1** Let \( q > d \) and assume that \( \mathcal{D} \) is Lipschitz. Let \( u^1 \) and \( u^2 \) be two solutions of the grade-two model with right-hand sides \( f^1 \) and \( f^2 \). If \( u^1 \in W^2_q(\mathcal{D})^d \) satisfies

\[ \| u^1 \|_{W^2_q(\mathcal{D})} \leq \eta/2\kappa, \]  

(6.3)

where \( \kappa \) is defined in (5.14), and \( u^2 \in H^2(\mathcal{D})^d \), then

\[ \eta |u^1 - u^2|_{H^1(\mathcal{D})} \leq 2 \| f^1 - f^2 \|_{H^{-1}(\mathcal{D})}. \]  

(6.4)

**Remark 6.2** Theorem 5.1 shows that, if there is a solution \( u_G \) for the grade-two fluid model satisfying the condition (6.3), then it must be unique.

The theorem means that the correspondence \( f \rightarrow u \) is locally Lipschitz continuous as a mapping of \( H^{-1}(\mathcal{D})^d \rightarrow H^1(\mathcal{D})^d \) provided we perturb around \( f^1 \in \mathcal{G}(W^2_q(\mathcal{D})^d \times W^1_q(\mathcal{D})) \), \( q > d \).

6.1 Proof of Theorem 6.1

Define the space \( \mathcal{W} = \{ v \in H^1_0(\mathcal{D})^d : \nabla \cdot v = 0 \text{ in } \mathcal{D} \} \). Denote by \( c \) the trilinear form applied to vector functions defined by

\[ c(u; v, w) = \int_\mathcal{D} (u(x) \cdot \nabla v(x)) \cdot w(x) \, dx = \sum_{i=1}^d \sum_{j=1}^d \int_\mathcal{D} u_i(x) \frac{\partial v_j(x)}{\partial x_i} \, w_j(x) \, dx. \]

Then for all \( u \in \mathcal{W} \) and for all \( v \in H^1(\mathcal{D})^d \),

\[ c(u; v, v) = 0. \]  

(6.5)

Similarly, define \( a(v, w) = \eta \int_\mathcal{D} \nabla v(x) : \nabla w(x) \, dx \). From [9, (5.1.3)] we know that

\[ \nabla \cdot (v \cdot \nabla A_1(v) + R(v)A_1(v) + A_1(v)R(v)^t) = (\text{curl } \Delta v) \times v + \nabla \rho(v), \]  

(6.6)

where \( \rho(v) = v \cdot \Delta v + \frac{1}{4} |A_1(v)|^2 \). We can thus characterize the solution \( u \) of the grade-two model equation (2.6) by the equation

\[ -\eta \Delta u + u \cdot \nabla u - \alpha (\text{curl } \Delta u) \times u + \nabla (p - \alpha \rho(u)) = f \quad \text{in } \mathcal{D}. \]  

(6.7)
Define \( w = u^1 - u^2 \) and \( q = p^1 - p^2 - \alpha (\rho(u^1) - \rho(u^2)) \). Then, in the sense of distributions,

\[
- \eta \Delta w + w \cdot \nabla u^1 + u^2 \cdot \nabla w - \alpha (\text{curl} \Delta w) \times u^1 - \alpha (\text{curl} \Delta u^2) \times w + \nabla q = f^1 - f^2 \text{ in } D.
\]

(6.8)

Now, we need to verify that \((\text{curl} \Delta w) \times u^1\) and \((\text{curl} \Delta u^2) \times w\) are in \(H^{-1}(D)^d\). Since the two terms are similar, it suffices to examine the first. For any \( w \in H^2(D)^d \), \((\text{curl} \Delta w) \times u^1\) is well defined in \(H^{-1}(D)^d\); thus consider \( z \times v\) with \( z \in H^{-1}(D)^d \) and \( v \) in a space to be determined. In the sense of distributions, we have

\[
< z \times v, \phi > = < z, v \times \phi > \quad \forall \phi \in C_0^\infty(D)^d,
\]

(6.9)

using the vector identity \((a \times b) \cdot c = a \cdot (b \times c)\). Therefore all we need is \( v \times \phi \in H^0_1(D)^d\) for any \( \phi \in H^0_1(D)^d\), and for this it suffices that \( v \) be in \( W^{1,q}_0(D)^d\) for some \( q > d\); then we have

\[
\|z \times v\|_{H^{-1}(D)} \leq \|z\|_{H^{-1}(D)} \|v\|_{W^{1,q}_0(D)}.
\]

Thus we can extend (6.9) by density to all \( v \in W^{1,q}_0(D)^d\) and \( \phi \in H^0_1(D)^d\). In particular, we conclude that, for all \( w \in H^2(D)^d\) and for all \( v \in W^{1,q}_0(D)^d\),

\[
< (\text{curl} \Delta w) \times v, v > = < \text{curl} \Delta w, v \times v > = 0.
\]

(6.10)

For \( d = 2, 3 \), this holds for all \( v \in (H^0_1(D) \cap H^2(D))^d\). Therefore, using (6.5) and (6.10), we have

\[
(f^1 - f^2, w) = a(w, w) + c(w, u^1, w) + c(u^2, w, w) - \alpha < (\text{curl} \Delta w) \times u^1, w >
\]

\[
- \alpha < (\text{curl} \Delta u^2) \times w, w >
\]

(6.11)

\[
= a(w, w) + c(w, u^1, w) - \alpha < (\text{curl} \Delta w) \times u^1, w >.
\]

From [9, (5.2.55), Lemma 5.2.16] we know that for all \( v \in H^2(D)^d\) and for all \( w \in H^0_1(D)^d \cap H^2(D)^d\), both with zero divergence,

\[
< \text{curl} (\Delta w) \times v, w > = \sum_{k=1}^d ((\nabla w)v_{k} \times (\nabla v)w_{k} - (\nabla v_{k})w_{k}w_{k}).
\]

(6.12)

Therefore (6.11) and (6.12) combine to yield

\[
a(w, w) = (f^1 - f^2, w) - c(w, u^1, w)
\]

\[
+ \alpha \sum_{k=1}^d ((\nabla w)u^1_{k} \times (\nabla u^1)w_{k} - (\nabla u^1_{k})w_{k}w_{k}).
\]

(6.13)

Thus (5.14) and (6.13) imply

\[
\eta |w|^2_{H^2(D)} \leq \kappa |u^1|^2_{W^2_0(D)}|w|^2_{H^1(D)} + \|f^1 - f^2\|_{H^{-1}(D)} |w|_{H^1(D)}.
\]

From the assumption (6.3), \( |u^1|^2_{W^2_0(D)} \leq \eta/2\kappa \), we find

\[
\frac{1}{2}\eta |w|^2_{H^2(D)} \leq \|f^1 - f^2\|_{H^{-1}(D)} |w|_{H^1(D)}.
\]

Dividing by \( |w|_{H^1(D)} \) and multiplying by 2 completes the proof of Theorem 6.1. QED
6.2 Proof of Theorem 5.1

We apply Theorem 6.1 with $u_1 = u_0$ and $u_2 = u_G$. Consulting (5.4) and (5.5), we see that $f_2 = f$ and

$$f_1 = f - \lambda_1^2 \nabla \cdot (u_0 \cdot \nabla M_0 + R(u_0)M_0 + M_0R(u_0)t).$$

Thus

$$\|f_1 - f_2\|_{H^{-1}(\mathcal{D})} \leq \lambda_1^2 \|u_0 \cdot \nabla M_0 + R(u_0)M_0 + M_0R(u_0)t\|_{L^2(\mathcal{D})}.$$  

Then (5.9) gives

$$\|f_1 - f_2\|_{H^{-1}(\mathcal{D})} \leq C\left(\frac{\lambda_1^2}{\eta^2}\|f\|_{W^3_0(\mathcal{D})}^3 + \frac{|\lambda_1|}{\eta}\|f\|_{W^2_0(\mathcal{D})}^2\right).$$

This proves (5.16). If (5.11) holds, then we conclude from (5.13) that

$$\eta|u_0 - u_G|_{H^1(\mathcal{D})} \leq 2C'\lambda_1^2\|f\|_{W^3_0(\mathcal{D})}\left(\|f\|_{W^3_0(\mathcal{D})} + K\right),$$

where $C'$ is an upper-bound for the constant expressions in (5.13). This proves (5.17).

Remark 6.3 From [9, page 191] we know that

$$((\text{curl } v) \times w)_i = w \cdot \nabla v - (v_i) \cdot w.$$  

Choosing $v = w$, we find

$$v \cdot \nabla v = (\text{curl } v) \times v + \frac{1}{2} \nabla|v|^2.$$  

Thus it is also possible to write the grade-two model as

$$-\eta \Delta u + \text{curl}(u - \alpha \Delta u) \times u + \nabla \pi = f \quad \text{in } \mathcal{D},$$

where $p = \pi - \frac{1}{2}|u|^2 + u \cdot \Delta u + \frac{1}{4}|A_1(u)|^2$. However, the change does not simplify the analysis.

7 Regularity for T

In addition to the results in section 3.1, we recall some additional results that allow us to

- prove (3.6),
- bootstrap the regularity for the Oldroyd solution, and
- prove further regularity results, in particular that $u_0 \in H^3(\mathcal{D})^d$.

To fit into the framework of [18], we view a general tensor $W$ as a function whose values are vectors of dimension $m$, and we use the Frobenius product “:” as the inner-product on such vectors, with norm $|W(x)| = \sqrt{W(x) : W(x)}$. In particular, [18, (4)] and [18, Theorem 3] can be phrased as follows.
Lemma 7.1 Suppose that \(2 \leq d \leq 4\), \(q \geq 2\), \(\mathcal{D} \subset \mathbb{R}^d\) is a bounded, Lipschitz domain, and \(\mathbf{v} \in H^1(\mathcal{D})^d\) with \(\nabla \cdot \mathbf{v} = 0\) in \(\mathcal{D}\) and \(\mathbf{v} \cdot \mathbf{n} = 0\) on \(\partial \mathcal{D}\). Suppose further that \(\mathbf{C}\) is an \(m \times m\) matrix-valued function such that \(\mathbf{C} \in L^\infty(\mathcal{D})^m\) and for some constant \(c_0 > 0\)
\[
(C(x)\xi) \cdot \xi \geq c_0 |\xi|^2 \quad \forall \xi \in \mathbb{R}^m \tag{7.1}
\]
for almost all \(x \in \mathcal{D}\). Then for all \(g \in L^q(\mathcal{D})^m\), there is a unique solution \(W \in L^q(\mathcal{D})^m\) of
\[
\mathbf{v} \cdot \nabla W + \mathbf{C}W = g, \tag{7.2}
\]
satisfying
\[
\|W\|_{L^q(\mathcal{D})} \leq \frac{1}{c_0} \|g\|_{L^q(\mathcal{D})}. \tag{7.3}
\]

As observed in [17], the results in [18] were stated for the special case when the size of the vector \(m\) was the same as the dimension of the domain \(d\) (i.e., \(m = d\)), but it can be easily checked that the result holds for vectors of arbitrary length \(m \geq 1\).

Using the techniques in [17], we can prove the following. We recall from (1.13) the Sobolev constant \(\sigma_q\).

Lemma 7.2 In addition to the conditions of Lemma 7.1, suppose that \(\mathbf{C} \in W^{1,q}(\mathcal{D})^m\) and \(\mathbf{v} \in W^{1,\infty}_\infty(\mathcal{D})^d\), satisfying
\[
\|\nabla \mathbf{v}\|_{L^\infty(\mathcal{D})} \leq \frac{1}{d} c_0, \quad \|\nabla \mathbf{C}\|_{L^q(\mathcal{D})} \leq \frac{1}{d} c_0 / \sigma_q, \quad q > d. \tag{7.4}
\]

Then for each \(g \in H^1(\mathcal{D})^m\), there is a unique solution \(W \in H^1(\mathcal{D})^m\) of (7.2) such that
\[
\int_\mathcal{D} |\nabla W|^2 \, dx \leq \frac{1}{c_0^2} \int_\mathcal{D} |g|^2 \, dx + \frac{4}{c_0^2} \int_\mathcal{D} |\nabla g|^2 \, dx. \tag{7.5}
\]

7.1 Proof of Lemma 7.2

Following [17], we introduce a regularized problem: find \(W^\epsilon \in H^1(\mathcal{D})^m\) such that
\[
-\epsilon \Delta W^\epsilon + \mathbf{v} \cdot \nabla W^\epsilon + \mathbf{C}W^\epsilon = g \quad \text{in} \quad \mathcal{D}, \tag{7.6}
\]
with natural boundary conditions as in (1.5), that is,
\[
\nabla (W^\epsilon)_{ij} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial \mathcal{D}, \quad \text{for} \quad i, j = 1, \ldots, d.
\]
Since the components of \(\mathbf{v}\) and \(\mathbf{C}\) belong to \(L^\infty(\mathcal{D})\), the bilinear form corresponding to the left-hand side of (7.6) is continuous, and it is checked below that it is elliptic. Therefore, (1.5) has a unique solution in \(H^1(\mathcal{D})^m\). To prove ellipticity, multiplying (7.6) by \((W^\epsilon)^J\)
integrated over \(\mathcal{D}\), and integrating by parts, we find
\[
\epsilon \int_\mathcal{D} |\nabla W^\epsilon|^2 \, dx + \int_\mathcal{D} (\mathbf{C}W^\epsilon) \cdot W^\epsilon \, dx + \int_\mathcal{D} (\mathbf{v} \cdot \nabla W^\epsilon) \cdot W^\epsilon \, dx = \int_\mathcal{D} g \cdot W^\epsilon \, dx. \tag{7.7}
\]
We have
\[
\int_\mathcal{D} (\mathbf{v} \cdot \nabla W^\epsilon) \cdot W^\epsilon \, dx = \sum_{ij} \int_\mathcal{D} (\mathbf{v} \cdot \nabla W^\epsilon_{ij}) W^\epsilon_{ij} \, dx = 0,
\]
since $\nabla \cdot v = 0$ in $\mathcal{D}$ and $v \cdot n = 0$ on $\partial \mathcal{D}$. From (7.1), we have
\[
\epsilon \int_{\mathcal{D}} |\nabla v|^2 \, dx + c_0 \int_{\mathcal{D}} |v|^2 \, dx \leq \left| \int_{\mathcal{D}} g \cdot W^e \, dx \right| \leq \| g \|_{L^2(\mathcal{D})} \| W^e \|_{L^2(\mathcal{D})}
\leq \frac{1}{2c_0} \| g \|_{L^2(\mathcal{D})}^2 + \frac{1}{2} c_0 \| W^e \|_{L^2(\mathcal{D})}^2.
\]

In particular, we obtain
\[
\| W^e \|_{L^2(\mathcal{D})} \leq \frac{1}{c_0} \| g \|_{L^2(\mathcal{D})}. \tag{7.8}
\]

Next, observing that $\nabla W^e$ belongs to $L^2(\mathcal{D})^m$ since $W^e \in H^1(\mathcal{D})^m$, we take the $L^2(\mathcal{D})^m$ inner-product of the terms on both sides of (7.6) with $-\Delta W^e$ and integrate by parts. Formally, this gives
\[
\epsilon \int_{\mathcal{D}} |\Delta W^e|^2 \, dx + \int_{\mathcal{D}} \nabla (v \cdot \nabla W^e) \cdot \nabla W^e \, dx
\]
\[
+ \int_{\mathcal{D}} \nabla (C W^e) \cdot \nabla W^e \, dx = \int_{\mathcal{D}} \nabla g \cdot \nabla W^e \, dx. \tag{7.9}
\]

But integration by parts requires that both $CW^e$ and $v \cdot \nabla W^e$ be in $H^1(\mathcal{D})^m$. This follows from the assumption (7.4). As in [17], we have the bound
\[
\left| \int_{\mathcal{D}} \nabla (v \cdot \nabla W^e) \cdot \nabla W^e \, dx \right| \leq \| \nabla W^e \|_{L^2(\mathcal{D})}^2 \| \nabla v \|_{L^\infty(\mathcal{D})}. \tag{7.10}
\]

Similarly, $\nabla (C W^e) \cdot \nabla W^e = (C \nabla W^e) \cdot \nabla W^e + (\nabla C) W^e \cdot \nabla W^e$, so that
\[
\int_{\mathcal{D}} \nabla (C W^e) \cdot \nabla W^e \, dx \geq c_0 \int_{\mathcal{D}} |\nabla W^e|^2 \, dx - \left| \int_{\mathcal{D}} (\nabla C) W^e \cdot \nabla W^e \, dx \right|. \tag{7.11}
\]

Thus (1.13), our assumption on $C$, and (7.8) imply
\[
\int_{\mathcal{D}} \nabla (C W^e) \cdot \nabla W^e \, dx \geq c_0 \int_{\mathcal{D}} |\nabla W^e|^2 \, dx - \sigma_2 \| C \|_{L^2(\mathcal{D})} \| W^e \|_{H^1(\mathcal{D})}^2
\geq c_0 \int_{\mathcal{D}} |\nabla W^e|^2 \, dx - \frac{1}{4} c_0 \| W^e \|_{H^1(\mathcal{D})}^2
\geq \frac{3}{4} c_0 \int_{\mathcal{D}} |\nabla W^e|^2 \, dx - \frac{1}{4} c_0 \| W^e \|_{L^2(\mathcal{D})}^2
\geq \frac{3}{4} c_0 \int_{\mathcal{D}} |\nabla W^e|^2 \, dx - \frac{1}{4} c_0 \| g \|_{L^2(\mathcal{D})}^2. \tag{7.12}
\]

Therefore (7.9) and (7.10) imply
\[
\epsilon \int_{\mathcal{D}} |\Delta W^e|^2 \, dx + \frac{3}{4} c_0 \int_{\mathcal{D}} |\nabla W^e|^2 \, dx \leq \| \nabla W^e \|_{L^2(\mathcal{D})}^2 \| \nabla v \|_{L^\infty(\mathcal{D})}
\]
\[
+ \frac{1}{4} c_0 \int_{\mathcal{D}} |g|^2 \, dx + \left| \int_{\mathcal{D}} \nabla g \cdot \nabla W^e \, dx \right|. \tag{7.13}
\]
Using our assumption on $v$, we get
\[ \epsilon \int_D |\Delta W^\epsilon|^2 dx + \frac{c_0}{4} \int_D |\nabla W^\epsilon|^2 dx \leq \frac{1}{4c_0} \int_D |g|^2 dx + \frac{1}{c_0} \int_D |\nabla g|^2 dx. \] (7.14)

Using (7.8), we conclude that $\| W^\epsilon \|_{H^1(D)}$ is also bounded independently of $\epsilon$. Thus there is a subsequence $\epsilon_j$ such that $W^{\epsilon_j}$ converges weakly to $\tilde{W} \in H^1(D)^m$, satisfying (7.5). The estimate (7.14) also shows that
\[ \epsilon \int_D |\Delta W^\epsilon|^2 dx \leq C \]
for some constant $C$ independent of $\epsilon$, and thus
\[ \| \epsilon \Delta W^\epsilon \|_{L^2(D)} \leq \sqrt{C} \epsilon \]
for all $\epsilon$. Taking the weak limit $\epsilon_j \to 0$ in (7.6) shows that $\tilde{W} \in H^1(D)^m$ is a solution of (7.2), and by uniqueness of such solutions, we conclude that the original solution $W$ must be in $H^1(D)^m$ and satisfy (7.5).

7.2 Proof of (3.6)

As an example, (3.5) is of the form (7.2) if we take $v = \lambda_1 u$ and (recall $\lambda_2 = \mu_2 = 0$)
\[ C\xi = \xi + \lambda_1 (R\xi + \xi R^t) - \mu_1 (E\xi + \xi E). \]

It may not appear at first that $C$ is in the form of a matrix function as written in (7.2), but this can be established as follows. As defined, for a.a. $x \in D$, $C$ is a linear operator on $\mathbb{R}^{d^2}$, given by
\[ \xi \to \xi + \lambda_1 (R(u(x))\xi + \xi R(u(x))^t) - \mu_1 (E(u(x))\xi + \xi E(u(x))). \]

But any linear operator on $\mathbb{R}^{d^2}$ can be represented by a matrix. By abuse of notation, we denote this matrix by $C(x)$. We can show that this is an integrable function, with
\[ \| C - \mathcal{I} \|_{L^\infty(D)} \leq 2(|\lambda_1| + |\mu_1|)\| \nabla u \|_{L^\infty(D)}. \] (7.15)

Moreover, we can show that
\[ \| \nabla C \|_{L^\infty(D)} \leq 2(|\lambda_1| + |\mu_1|)\| u \|_{W^{1,2}_q(D)}. \] (7.16)

Thus (7.15) shows that (7.1) is satisfied if we define
\[ c_0 = 1 - 2(|\lambda_1| + |\mu_1|)\| \nabla u \|_{L^\infty(D)}, \]
provided of course that $c_0 > 0$. Suppose that we assume
\[ \| \nabla u \|_{L^\infty(D)} \leq \frac{1}{4 + 2(|\lambda_1| + |\mu_1|)}. \] (7.17)

Then
\[ c_0 \geq (1 + \frac{1}{2}(|\lambda_1| + |\mu_1|))^{-1}, \]
and the first condition in (7.4) is satisfied. The second condition in (7.4) is satisfied if
\[
\| \nabla C \|_{L^q(D)} \leq \frac{c_0}{4\sigma_q} \left( 1 - 2(|\lambda_1| + |\mu_1|) \| \nabla u \|_{L^\infty(D)} \right).
\]
In view of (7.16), we need to prove that
\[
2(|\lambda_1| + |\mu_1|) \| u \|_{W^2_q(D)} \leq \frac{1}{4\sigma_q} \left( 1 - 2(|\lambda_1| + |\mu_1|) \| \nabla u \|_{L^\infty(D)} \right).
\]
Note that
\[
(1 - 2(|\lambda_1| + |\mu_1|) \| \nabla u \|_{L^\infty(D)}) \geq (1 - 2c_q(|\lambda_1| + |\mu_1|) \| u \|_{W^2_q(D)}).
\]
Thus the second condition in (7.4) holds if
\[
2(|\lambda_1| + |\mu_1|) \| u \|_{W^2_q(D)} \leq \frac{1}{4\sigma_q} \left( 1 - 2c_q(|\lambda_1| + |\mu_1|) \| u \|_{W^2_q(D)} \right).
\]
This is equivalent to
\[
\| u \|_{W^2_q(D)} \leq \frac{1}{(4\sigma_q + c_q)2(|\lambda_1| + |\mu_1|)}.
\]
Then Lemma 7.1 and Lemma 7.2 prove (3.6).

### 7.3 Higher derivatives

This process can be iterated, although the form of the transport equation changes at each step. Thus we let \( W = \nabla T \) and consider the equation that it solves. First, assume that \( u \) is sufficiently smooth and that \( T \) solves
\[
T + \lambda_1 (u \cdot \nabla T + RT + TR^t) - \mu_1 (ET + TE) = \eta(\nabla u + \nabla u^t), \quad \text{(7.18)}
\]
where \( R = \frac{1}{2} (\nabla u^t - \nabla u) \). Applying Lemma 3.2, we have
\[
\begin{align*}
\| T \|_{L^q(D)} & \leq (2 + |\lambda_1| + |\mu_1|)\eta \| \nabla u \|_{L^q(D)}, \\
\| T \|_{H^1(D)} & \leq 2(2 + |\lambda_1| + |\mu_1|)\eta \| \nabla u \|_{H^1(D)}, \quad \text{(7.19)}
\end{align*}
\]
provided (3.4) is satisfied.

Now we derive a bound for \( \nabla^2 T \). Recall that \( u \cdot \nabla T = (\nabla T)u \). Then
\[
\nabla (u \cdot \nabla T) = u \cdot \nabla (\nabla T) + \nabla T \nabla u
\]
and \( W = \nabla T \) solves
\[
W + \lambda_1 (u \cdot \nabla W + W \nabla u + RW + WR^t) - \mu_1 (E W + EW) = g \quad \text{(7.20)}
\]
where
\[
g = \eta(\nabla^2 u + \nabla^2 u^t) - \lambda_1 ((\nabla R)T + T \nabla R^t) + \mu_1 ((\nabla E)T + T \nabla E^t). \quad \text{(7.21)}
\]
Thus the operator \( C \) in this case is defined by
\[
C \xi = \xi + \lambda_1 (\xi \nabla u + \mathbf{R} \xi + \xi \mathbf{R}^t) - \mu_1 (E \xi + \xi E),
\]
and \( \mathbf{v} = \lambda_1 \mathbf{u} \). Using similar arguments as in section 7.2, we define
\[
c_0 = 1 - 2(|\lambda_1| + |\mu_1|) \| \nabla \mathbf{u} \|_{L^\infty(D)}^{2}
\]
and assume
\[
\| \nabla \mathbf{u} \|_{L^\infty(D)} \leq \frac{1}{4 + 2(|\lambda_1| + |\mu_1|)}, \quad \| \mathbf{u} \|_{W^2_q(D)} \leq \frac{1}{(4\sigma_q + c_q)2(|\lambda_1| + |\mu_1|)}.
\]
(7.22)
so that \( c_0 \geq \left( 1 + \frac{1}{2}(|\lambda_1| + |\mu_1|) \right)^{-1} \). Applying Lemma 7.2, we conclude that
\[
\int_D |\nabla^2 \mathbf{T}|^2 \, dx \leq \left( 1 + \frac{1}{2}(|\lambda_1| + |\mu_1|) \right)^2 \left( \int_D |\mathbf{g}|^2 \, dx + 4 \int_D |\nabla \mathbf{g}|^2 \, dx \right).
\]
(7.23)
Using the Sobolev inequality (1.11), we get from (7.19) and (7.21) that
\[
\| \mathbf{g} \|_{L^2(D)} \leq 2\eta \| \mathbf{u} \|_{H^2(D)} + 2(|\lambda_1| + |\mu_1|) \| \nabla^2 \mathbf{u} \|_{L^2(D)} \nonumber \\
\leq 2\eta \| \mathbf{u} \|_{H^2(D)} + 2\sigma_q(|\lambda_1| + |\mu_1|) \| \mathbf{u} \|_{H^2(D)} \| \mathbf{T} \|_{W^3_q(D)}. \nonumber
\]
(7.24)
Taking the gradient of (7.21), we get
\[
\| \nabla \mathbf{g} \|_{L^2(D)} \leq 2\eta \| \mathbf{u} \|_{H^3(D)} \nonumber \\
+ 2(|\lambda_1| + |\mu_1|) \left( \| \nabla^2 \mathbf{u} \|_{L^2(D)} + \| \nabla^2 \mathbf{u} \|_{L^2(D)} \right). \nonumber
\]
(7.25)
Using the Sobolev inequality (1.12) again, we get
\[
\| \nabla^2 \mathbf{u} \|_{L^2(D)} \leq \sigma_q \| \mathbf{u} \|_{H^3(D)} \| \nabla \mathbf{T} \|_{L^2(D)}. \nonumber
\]
Similarly, the Sobolev inequality (1.11) implies
\[
\| \nabla^3 \mathbf{u} \|_{L^2(D)} \leq c_q \| \mathbf{u} \|_{H^3(D)} \| \mathbf{T} \|_{W^3_q(D)}. \nonumber
\]
Thus (7.25) becomes
\[
\| \nabla \mathbf{g} \|_{L^2(D)} \leq \left( 2\eta + 2(\sigma_q + c_q)(|\lambda_1| + |\mu_1|) \| \mathbf{T} \|_{W^3_q(D)} \right) \| \mathbf{u} \|_{H^3(D)}.
\]
(7.26)
Combining (7.24) and (7.26), we get
\[
\| \mathbf{g} \|_{L^2(D)}^2 + 4 \| \nabla \mathbf{g} \|_{L^2(D)}^2 \leq 20 \left( \eta + (\sigma_q + c_q)(|\lambda_1| + |\mu_1|) \| \mathbf{T} \|_{W^3_q(D)} \right) \| \mathbf{u} \|_{H^3(D)}. \nonumber
\]
(7.27)
Applying (7.27) to (7.23), we get
\[
\| \nabla^2 \mathbf{T} \|_{L^2(D)} \leq 5 \left( 1 + \frac{1}{2}(|\lambda_1| + |\mu_1|) \right) \left( \eta + (\sigma_q + c_q)(|\lambda_1| + |\mu_1|) \| \mathbf{T} \|_{W^3_q(D)} \right) \| \mathbf{u} \|_{H^3(D)}. \nonumber \quad (7.28)
\]
We summarize this in the following bootstrapping lemma.

**Lemma 7.3** Suppose that the assumptions of Lemmas 7.1 and 7.2 hold. Suppose that \( \mathbf{u} \in H^3(D)^d \) satisfies (7.22) and \( \mathbf{T} \in W^1_q(D)^{d^2} \) solves (7.18). Then \( \mathbf{T} \in H^2(D)^{d^2} \) and the bound (7.28) holds.

This says that, if \( \mathbf{u} \) is smooth, then \( \mathbf{T} \) is smooth. But there is not a simple relationship that bounds \( \mathbf{u} \) in terms of \( \mathbf{T} \) for the 3-parameter Oldroyd model. Thus to get a bound on \( \| \mathbf{u} \|_{H^3(D)} \) in terms of \( \| \mathbf{T} \|_{W^3_q(D)} \), we must reach back into the arguments in [17].
7.4 Solution algorithm

The following algorithm is used in [17] to demonstrate existence for the Oldroyd model. Given \( u^{n-1}, T^{n-1}, p^{n-1} \), we define \( u^n, T^n, p^n \) as follows. First we solve

\[
-\eta \Delta u^n + u^n \cdot \nabla u^n + \nabla \pi^n = F(f, u^{n-1}, p^{n-1}, T^{n-1}) \quad \text{in } D,
\]
\[
\nabla \cdot u^n = 0 \quad \text{in } D, \quad u^n = 0 \quad \text{on } \partial D
\]  

(7.29)

to determine \( u^n \) and \( \pi^n \), where \( F \) is defined by

\[
F(f, u, p, T) = f + \lambda_1 u \cdot \nabla f + \lambda_1 (\nabla u)\cdot \nabla p - \lambda_1 (u \cdot \nabla (u \cdot \nabla u)) - \nabla ((\nabla u) T)
\]
\[
- \lambda_2 \nabla \cdot (ET + TE).
\]

(7.30)

Then we solve

\[
p^n + \lambda_1 u^n \cdot \nabla p^n = \pi^n
\]

(7.31)
to determine \( p^n \). We continue the notation

\[
E^n = \frac{1}{2} (\nabla u^n + (\nabla u^n)^t) \quad \text{and} \quad R^n = \frac{1}{2} (-\nabla u^n + (\nabla u^n)^t).
\]

Finally, we solve

\[
T^n + \lambda_1 (u^n \cdot \nabla T^n + R^n T^n + T^n (R^n)^t) - \mu_1 (E^n T^n + T^n E^n) = 2\eta E^n.
\]

(7.32)

More precisely, we first solve the Navier-Stokes equations (7.29) for \( u^n \in W_q^2(D)^d \) and \( \pi^n \in W_q^1(D) \). Then we solve the scalar transport equation (7.31) for \( p^n \in \mathcal{W}_q^1(D) \). Finally, we solve (7.32) for \( T^n \in \mathcal{W}_q^1(D)^{d^2} \). We begin the iteration with \( u^0 = 0, p^0 = 0 \) and \( T^0 = 0 \).

It is proved in [17] that if

\[
\eta \| u^n \|_{W_q^2(D)} + \| \pi^n \|_{W_q^1(D)} \leq \gamma
\]

(7.33)
and \( \gamma \leq C_0 \) and \( \| f \|_{W_q^1(D)} \leq \phi \), then

\[
\eta \| u^{n+1} \|_{W_q^2(D)} + \| \pi^{n+1} \|_{W_q^1(D)} \leq C_{q,D,\eta}(1 + C_{1} \gamma) \phi + C_2 \gamma^2 + C_3 \gamma^3.
\]

(7.34)

Thus choosing

\[
\gamma \leq \gamma_1 := \min \left\{ 1, C_0, C_1^{-1}, \frac{1}{4} C_2^{-1}, \frac{1}{2} C_3^{-1/2} \right\}
\]

(7.35)
we conclude that

\[
\eta \| u^{n+1} \|_{W_q^2(D)} + \| \pi^{n+1} \|_{W_q^1(D)} \leq 2C_{q,D,\eta} \phi + \frac{1}{2} \gamma \leq \gamma,
\]

(7.36)
provided that

\[
\phi \leq \frac{\gamma}{4C_{q,D,\eta}}.
\]

(7.37)

Note that we can take \( \gamma \) as small as we want, subject to (7.35) \( \gamma \leq \gamma_1 \) and the restriction

\[
\| f \|_{W_q^1(D)} = \phi \leq \gamma/(4C_{q,D,\eta}).
\]

Similarly, [17] proves

\[
\| T^n \|_{W_q^1(D)} + \| p^n \|_{W_q^1(D)} \leq \hat{\sigma}(\eta \| u^n \|_{W_q^2(D)} + \| \pi^n \|_{W_q^1(D)}).
\]

(7.39)
The constant $\hat{\sigma}$ is given in [17, Lemma 6.1]. By induction, we conclude that, for all $n$,

$$\eta \| u^n \|_{W^2_q(D)} + \| \pi^n \|_{W^1_q(D)} \leq \gamma$$

$$\| T^n \|_{W^1_q(D)} + \| p^n \|_{W^1_q(D)} \leq \hat{\sigma} \gamma,$$

(7.38)

for any $\gamma$ bounded by (7.35) and any $\phi$ bounded by (7.37). The strong convergence

$$(u^n, p^n, T^n) \to (u, p, T)$$

in $H^1(D)^d \times L^2(D) \times L^2(D)^d$ is also proved in [17] under these restrictions.

### 7.5 Higher norms

Now let us consider bounds in stronger spaces. To simplify notation, define

$$B = \{ (u, p, T) : \eta \| u \|_{W^2_q(D)} \leq \gamma, \| T \|_{W^1_q(D)} + \| p \|_{W^1_q(D)} \leq \hat{\sigma} \gamma \}. \quad (7.39)$$

Note that for some constant $c > 0$,

$$\| u^n \cdot \nabla u^n \|_{H^1(D)} \leq c \gamma^2 / \eta^2. \quad (7.40)$$

In view of (1.10), and the fact that $F(f, u^0, p^0, T^0) = F(f, 0, 0, 0) = f$, we have

$$\eta \| u^1 \|_{H^3(D)} + \| \pi^1 \|_{H^2(D)/\mathbb{R}} \leq C_3 \| f - u^1 \cdot \nabla u^1 \|_{H^1(D)} \leq C_3 (\phi + c \gamma^2 / \eta^2), \quad (7.41)$$

where we recall our assumption $|D| \leq 1$ which implies $\| f \|_{H^1(D)} \leq \| f \|_{W^1_q(D)}$. So we can assume that $u^n \in H^3(D)^d$ and $\pi^n \in H^2(D)/\mathbb{R}$, at least for $n = 1$.

Applying (7.28) and the bounds (7.38), we find

$$\| \nabla^2 T^n \|_{L^2(D)} \leq C_T \| u^n \|_{H^3(D)},$$

$$C_T = 5 \left( 1 + \frac{1}{2} (|\lambda_1| + |\mu_1|) \right) (\eta + (\sigma_q + c_q) (|\lambda_1| + |\mu_1|) \hat{\sigma} \gamma), \quad (7.42)$$

provided that $\gamma$ is chosen sufficiently small so that $u^n$ satisfies (7.22). This means that we need to assume the following. The first term in (7.22) can be estimated by

$$\| \nabla u \|_{L^\infty(D)} \leq c_q \gamma \leq \frac{1}{4 + 2 (|\lambda_1| + |\mu_1|)},$$

provided that

$$\gamma \leq \frac{1}{c_q (4 + 2 (|\lambda_1| + |\mu_1|))}.$$  

Thus (7.22) is satisfied for $u^n$ if

$$\gamma \leq \gamma_2 := \min \left\{ \gamma_1, \frac{1}{c_q (4 + 2 (|\lambda_1| + |\mu_1|))}, \left( \frac{1}{4 \sigma_q + c_q} \right) \frac{1}{2 (|\lambda_1| + |\mu_1|)} \right\}, \quad (7.43)$$

where $\gamma_1$ is defined in (7.35). Therefore, for such $\gamma$, (7.42) holds.

Using the same ideas leading to (7.28), we can also show that

$$\| p^n \|_{H^2(D)/\mathbb{R}} \leq C_\pi \| \pi^n \|_{H^2(D)/\mathbb{R}}. \quad (7.44)$$
To prove this, observe that, for any \( u \) and \( p \),
\[
\nabla (u \cdot \nabla p) = (\nabla u)^t \nabla p + u \cdot \nabla (\nabla p).
\]

Taking the gradient of (7.31), we see that \( W = \nabla p^n \) solves
\[
W + \lambda_1 u^n \cdot \nabla W + \lambda_1 (\nabla u^n)^t W = \nabla \pi^n.
\]

This is (7.2) with \( C = I + \lambda_1 (\nabla u^n)^t \) and \( g = \nabla \pi^n \). Define \( c_0 = 1/(1 + \lambda_1) \). Using (1.11) and (7.38), we find
\[
\| \nabla u^n \|_{L^\infty(\mathcal{D})} \leq c_q \| u^n \|_{W^2_q(\mathcal{D})} \leq \frac{c_q \gamma}{\eta},
\]
so we have \( \| \nabla u^n \|_{L^\infty(\mathcal{D})} \leq \frac{1}{4} c_0 \), as required by Lemma 7.2, provided that
\[
\gamma \leq \frac{\eta c_0}{4 c_q} = \frac{\eta}{4 c_q (1 + |\lambda_1|)}.
\]
(7.45)

Then for \( \xi \neq 0 \),
\[
|\xi|^{-2} (C(x) \xi) \cdot \xi \geq 1 - |\lambda_1| \| \nabla u \|_{L^\infty(\mathcal{D})} \geq 1 - \frac{1}{4} |\lambda_1| c_0 = 1 - \frac{|\lambda_1|}{4 (1 + |\lambda_1|)}
\]
\[
= \frac{4 (1 + |\lambda_1|) - |\lambda_1|}{4 (1 + |\lambda_1|)} = \frac{4 + 3 |\lambda_1|}{4 (1 + |\lambda_1|)} = c_0 + \frac{3 |\lambda_1|}{4 (1 + |\lambda_1|)} \geq c_0,
\]
(7.46)
as required by Lemma 7.1. Note that
\[
\| \nabla C \|_{L^2(\mathcal{D})} = |\lambda_1| \| \nabla^2 u^n \|_{L^2(\mathcal{D})} \leq \frac{|\lambda_1| \gamma}{\eta} \leq \frac{1}{4} c_0 / c_q,
\]
as required by Lemma 7.2, provided that
\[
\gamma \leq \frac{\eta c_0}{4 c_q |\lambda_1|} = \frac{\eta}{4 c_q |\lambda_1| (1 + |\lambda_1|)}.
\]
(7.47)

With the added constraints (7.45) and (7.47) on \( \gamma \), Lemma 7.2 implies that
\[
\| \nabla^2 p^n \|_{L^2(\mathcal{D})}^2 \leq 4 (1 + |\lambda_1|)^2 \| \pi^n \|_{H^2(\mathcal{D})}^2.
\]

Thus (7.44) holds with
\[
C_\pi = \sqrt{\gamma^2 + 4 (1 + |\lambda_1|)^2},
\]
(7.48)
provided that
\[
\gamma \leq \gamma_3 := \min \left\{ \gamma_2, \frac{\eta}{4 c_q (1 + |\lambda_1|)}, \frac{\eta}{4 c_q |\lambda_1| (1 + |\lambda_1|)} \right\},
\]
(7.49)
where \( \gamma_2 \) is defined in (7.43).
7.6 Estimating $\mathcal{F}$ in $H^1$

Now we estimate $\nabla \mathcal{F}(f, u^n, p^n, T^n)$ in $L^2(\mathcal{D})^d$. To do so, we examine the terms in (7.30). Suppose that $f \in H^2(\mathcal{D})^d$. We then have

$$\| \nabla (f + \lambda_1 u^n \cdot \nabla f) \|_{L^2(\mathcal{D})} \leq (1 + \beta_1 \gamma) \phi + \beta_1 \gamma \| f \|_{H^2(\mathcal{D})}, \quad \beta_1 = \eta^{-1} c_q |\lambda_1|. \quad (7.50)$$

where $c_q$ is the constant in the Sobolev inequality (1.11) and $\phi$ satisfies (7.37). Next,

$$\| \nabla ((\nabla u^n)' \cdot p^n) \|_{L^2(\mathcal{D})} \leq \eta^{-1} (\sigma_q + c_q) \| p^n \|_{H^2(\mathcal{D})/\mathbb{R}}, \quad (7.51)$$

where $\sigma_q$ is the constant in the Sobolev inequality (1.12). To estimate the subsequent term in (7.30), we need to clarify the tensor quantities. Thus

$$u \cdot \nabla (u \cdot \nabla u) = [\nabla ((\nabla u)u)] u.$$

Note that $\nabla ((\nabla u)u)$ is a matrix, given by

$$[\nabla ((\nabla u)u)]_{ik} = \sum_j (u_{i,j}u_j)_{ik} = \sum_j (u_{i,jk}u_j + u_{i,j}u_j, k) = u \cdot \nabla u_{ik} + ((\nabla u)^2)_{ik}.$$

Thus

$$\nabla ((\nabla u)u) = u \cdot \nabla (\nabla u) + (\nabla u)^2 = (\nabla^2 u)u + (\nabla u)^2.$$

Note that $\nabla^2 u$ is a tensor of arity 3, so that $(\nabla^2 u)u$ is a matrix. Therefore

$$u \cdot \nabla (u \cdot \nabla u) = ((\nabla^2 u)u + (\nabla u)^2) u.$$

Differentiating, we find

$$\nabla (u \cdot \nabla (u \cdot \nabla u)) = ((\nabla^2 u)u + (\nabla u)^2) \nabla u + ((\nabla^3 u)u + (\nabla^2 u)\nabla u + 2(\nabla u)(\nabla^2 u)) u.$$

Recall that $\| u^n \|_{W^3(\mathcal{D})} \leq c_q \gamma / \eta$. Therefore

$$\| \nabla ((u^n \cdot \nabla (u^n \cdot \nabla u^n)) \|_{L^2(\mathcal{D})} \leq 6 \left( \frac{c_q \gamma}{\eta} \right)^2 \| u^n \|_{H^3(\mathcal{D})} = \beta_2 \gamma \| u^n \|_{H^3(\mathcal{D})}, \quad \beta_2 = \frac{6c_q^2 \gamma}{\eta^2}. \quad (7.52)$$

Next, we have

$$\nabla \cdot ((\nabla u) T) = \nabla^2 u : T + (\nabla u) \nabla \cdot T,$$

where $(\nabla^2 u : T)_i = \sum_{jk} u_{i,jk} T_{jk}$. Therefore

$$\nabla \nabla \cdot ((\nabla u) T) = \nabla^3 u : T + \nabla^2 u : \nabla T + (\nabla^2 u) \nabla \cdot T + (\nabla u) \nabla \nabla \cdot T,$$

The colon multiplication notation requires some further specification. For example, the colon operation always indicates summation over two indices, but which indices are used for a tensor of arity 3 or higher must be specified. In particular,

$$(\nabla (\nabla^2 u : T))_{ij} = \left( \sum_{jk} u_{i,jk} T_{jk} \right)_{ij} = \left( \sum_{jk} u_{i,jkl} T_{jk} + u_{i,jk} T_{jkl} \right)_{ij} := (\nabla^3 u : T + \nabla^2 u : \nabla T)_{ij}. \quad (7.53)$$
However, the indices chosen do not affect the norm estimates. For example,
\[
\| \nabla^3 u : T \|_{L^2(D)}^2 = \sum_u \| (\nabla^3 u : T)_u \|_{L^2(D)}^2 = \sum_{ijkl} \| u_{ijkl} T_{jk} \|_{L^2(D)}^2 
\leq \sum_{ijkl} \| u_{ijkl} \|_{L^2(D)}^2 \| T_{jk} \|_{L^\infty(D)}^2 \leq \| \nabla^3 u \|_{L^2(D)}^2 \| T \|_{L^\infty(D)}^2.
\]
Thus (1.4), (1.11), (1.12), and (7.42) yield
\[
\| \nabla \nabla \cdot ((\nabla u^n)T^n) \|_{L^2(D)} \leq \| u^n \|_{H^3(D)} \left( \| T^n \|_{L^\infty(D)} + (1 + d)\| T^n \|_{W_0^3(D)} \right) 
+ \| u^n \|_{W_0^3(D)} \| \nabla \nabla \cdot T^n \|_{L^2(D)} 
\leq \| u^n \|_{H^3(D)} (c_q + 1 + d)\| T^n \|_{W_0^3(D)} + c_q d \| u^n \|_{W_0^3(D)} \| \nabla^2 T^n \|_{L^2(D)} 
\leq \| u^n \|_{H^3(D)} \left( (c_q + 1 + d)\sigma \gamma + \frac{c_q d}{\eta} C_T \right) = \beta_3 \| u^n \|_{H^3(D)},
\]
\[
\beta_3 = (c_q + 1 + d)\sigma \gamma + \frac{c_q d}{\eta} C_T.
\]
Finally, \( \nabla \cdot (TU) = \nabla T : U + TV \cdot U \), where \( (\nabla T : U)_i = \sum_{jk} T_{ij} U_{jk} \). Therefore
\[
\nabla \nabla \cdot (TU) = \nabla^2 T : U + \nabla T \cdot \nabla U + T \nabla \nabla \cdot U.
\]
Thus (1.11), (1.12), and (7.42) imply
\[
\frac{1}{2} \| \nabla \nabla \cdot ((E^n T^n + T^n E^n)) \|_{L^2(D)} \leq \| T^n \|_{H^2(D)} \| \nabla u^n \|_{L^\infty(D)} 
+ (1 + d)\sigma_q \| \nabla T^n \|_{L^2(D)} \| u^n \|_{H^3(D)} + d \| T^n \|_{L^\infty(D)} \| u^n \|_{H^3(D)} 
\leq \left( \frac{C_T c_q \gamma}{\eta} + (1 + d)\sigma_q \sigma \gamma + dc_q \sigma \gamma \right) \| u^n \|_{H^3(D)} \]
\[
\beta_4 \| u^n \|_{H^3(D)}, \quad \beta_4 = \frac{C_T c_q}{\eta} + (\sigma_q + d(\sigma_q + c_q))\sigma \gamma.
\]
Combining (7.50), (7.51), (7.52), (7.53), and (7.54), we find
\[
\| \nabla F(f, u^n, p^n, T^n) \|_{L^2(D)} \leq (1 + \beta_1 \gamma) \phi + \beta_1 \gamma \| f \|_{H^2(D)} 
+ \eta^{-1} \gamma |\lambda_1| (\sigma_q + c_q) \| p^n \|_{H^2(D)} + \gamma |\lambda_1| (\beta_2 + \beta_3) \| u^n \|_{H^3(D)} \]
\[
+ \gamma |\lambda_1 - \mu_1| \beta_4 \| u^n \|_{H^3(D)}.
\]
Re-arranging and applying (7.44), we find
\[
\| \nabla F(f, u^n, p^n, T^n) \|_{L^2(D)} \leq \phi + \gamma \left( \beta_1 (\phi + \| f \|_{H^2(D)}) 
+ C_x \| \pi^n \|_{H^2(D)/\mathbb{R}} + C_u \| u^n \|_{H^3(D)} \right),
\]
where \( C_x \) is defined in (7.44) and
\[
C_u = |\lambda_1| (\beta_2 + \beta_3) + |\lambda_1 - \mu_1| \beta_4.
\]
In [17], the bound
\[
\| F(f, u^n, p^n, T^n) \|_{L^2(D)} \leq (1 + C_f \gamma) \phi + C' \gamma^2 + C'' \gamma^3
\]
(7.57)
is proved, where
\[ C' = 6 C f \hat{\sigma}, \quad \text{and} \quad C'' = 2 C f \sigma_q / \eta^2. \]
The constant \( \hat{\sigma} \) is given in [17, Lemma 6.1]. Then (7.57) and (7.56) imply
\begin{align*}
\|F(f, u^n, p^n, T^n)\|_{H^1(D)} & \leq 2 \phi + \gamma \left( C_f (2 \phi + \| f \|_{H^1(D)}) + |\lambda_1| \beta_1 + C' \gamma + C'' \gamma^2 \right. \\
& \quad + C_u \| \pi^n \|_{H^2(D) / \mathbb{R}} + C_u \| u^n \|_{H^3(D)} \bigg) . \tag{7.58}
\end{align*}

### 7.7 Induction step

In view of (7.29), (1.10) and (7.40), we have
\[ \eta \| u^{n+1} \|_{H^1(D)} + \| \pi^{n+1} \|_{H^2(D)/\mathbb{R}} \leq C_3 \left( \| F(f, u^n, p^n, T^n) \|_{H^1(D)} + c \gamma^2 / \eta^2 \right). \tag{7.59} \]
Then (7.58) and (7.59) imply
\begin{align*}
\eta \| u^{n+1} \|_{H^1(D)} + \| \pi^{n+1} \|_{H^2(D)/\mathbb{R}} & \leq C_3 \gamma \left( C_3 \gamma \| \pi^n \|_{H^2(D)} + C_u \| u^n \|_{H^3(D)} \right) + K, \\
K & = C_3 \left( 2 \phi + \gamma \left( c \gamma / \eta^2 + C_f (2 \phi + \| f \|_{H^1(D)}) + |\lambda_1| \beta_1 + C' \gamma + C'' \gamma^2 \right) \right), \tag{7.60}
\end{align*}
so we conclude by induction from (7.41) that \( u^n \in H^3(D)^d \) and \( \pi^n \in H^2(D) \) for all \( n \). Now assume that
\[ \gamma \leq \gamma_4 := \min \left\{ \gamma_3, \eta \left( 2 C_3 \max \{ \eta C_3, C_u \} \right)^{-1} \right\}, \tag{7.61} \]
where \( \gamma_3 \) is defined in (7.49). Then
\[ \eta \| u^{n+1} \|_{H^1(D)} + \| \pi^{n+1} \|_{H^2(D)/\mathbb{R}} \leq \frac{1}{2} \left( \| \pi^n \|_{H^2(D)/\mathbb{R}} + \eta \| u^n \|_{H^3(D)} \right) + K. \tag{7.62} \]
By induction, we conclude from (7.41) that, for all \( n \),
\[ \eta \| u^n \|_{H^1(D)} + \| \pi^n \|_{H^2(D)/\mathbb{R}} \leq 2 K. \tag{7.63} \]
Thus there is a subsequence \( u^{n_j} \) that converges weakly in \( H^3(D)^d \) to some \( u \in H^3(D)^d \). But since this subsequence converges to \( u_0 \) in \( H^1(D)^d \), we must have \( u = u_0 \), and thus \( u_0 \in H^3(D)^d \) and
\[ \| u_0 \|_{H^3(D)} \leq 2 K. \tag{7.64} \]
Thus we have proved the following.

**Theorem 7.4** Suppose that the assumptions of Lemma 3.1 hold and that \( \gamma \leq \gamma_4 \) (defined in (7.61)), and assume that (1.10) holds. Then there is a constant \( \phi > 0 \) (satisfying (7.37)) such that, if \( \| f \|_{W^2_0(D)} \leq \phi \) and \( f \in H^2(D)^d \), then the solution \( u_0 \) of (1.1) and (3.2) guaranteed by Lemma 3.1 is in \( H^3(D)^d \). Moreover, \( u_0 \) satisfies the bound (7.64), where \( K \) is given in (7.60), and thus
\[ \| u_0 - u_G \|_{H^1(D)} \leq C \lambda_1^2 \]
as \( \lambda_1 \to 0 \).
7.8 Stress convergence

It is also of interest to consider the convergence of the stresses as \( \alpha = \eta \lambda \to 0 \). Define \( \mathcal{E} = T_o - T_g \) and \( \mathbf{w} = \mathbf{u}_o - \mathbf{u}_g \).

From (2.2), we have when \( \alpha_1 + \alpha_2 = 0 \), setting \( \alpha = \alpha_1 \),

\[
T_g = \eta ((\nabla \mathbf{u}_g)^t + \nabla \mathbf{u}_o) + \alpha \mathcal{G}(\mathbf{u}_g, \mathbf{A}_g, 0) = \eta \mathbf{A}_g + (\alpha/\eta)\mathcal{G}(\mathbf{u}_g, \eta \mathbf{A}_g, 0),
\]

where for simplicity we write \( \mathbf{A}_x = \nabla \mathbf{u}_x + \nabla \mathbf{u}_x^t \) for \( x = g \) or \( x = o \).

On the other hand, when \( \mu_1 = 0 \) we have from (3.2) that

\[
T_o + \lambda_1 \mathcal{G}(\mathbf{u}_o, T_o, 0) = \eta \mathbf{A}_o.
\]

If we set \( \alpha = -\eta \lambda_1 \), we get

\[
T_g + \lambda_1 \mathcal{G}(\mathbf{u}_g, \eta \mathbf{A}_g, 0) = \eta \mathbf{A}_g.
\]

Subtracting these two expressions, we get

\[
\mathcal{E} + \lambda_1 (\mathcal{G}(\mathbf{u}_o, T_o, 0) - \mathcal{G}(\mathbf{u}_g, \eta \mathbf{A}_g, 0)) = \eta (\mathbf{A}_o - \mathbf{A}_g).
\]

Adding and subtracting, we find

\[
\mathcal{G}(\mathbf{u}_o, T_o, 0) - \mathcal{G}(\mathbf{u}_g, \eta \mathbf{A}_g, 0) = \mathcal{G}(\mathbf{u}_o, T_o, 0) - \mathcal{G}(\mathbf{u}_o, T_g, 0) + \mathcal{G}(\mathbf{u}_o, T_g, 0) - \mathcal{G}(\mathbf{u}_o, \eta \mathbf{A}_g, 0) - \mathcal{G}(\mathbf{u}_o, \eta \mathbf{A}_g, 0) + \mathcal{G}(\mathbf{u}_o, \eta \mathbf{A}_g, 0)
\]

\[
= \mathcal{G}(\mathbf{u}_o, \mathcal{E}, 0) + \mathcal{G}(\mathbf{u}_o, T_g - \eta \mathbf{A}_g, 0) + \mathcal{G}(\mathbf{u}_o - \mathbf{u}_g, \eta \mathbf{A}_g, 0)
\]

\[
= \mathcal{G}(\mathbf{u}_o, \mathcal{E}, 0) - \lambda_1 \mathcal{G}(\mathbf{u}_o, \mathcal{G}(\mathbf{u}_g, \eta \mathbf{A}_g, 0), 0) + \mathcal{G}(\mathbf{u}_o - \mathbf{u}_g, \eta \mathbf{A}_g, 0).
\]

Therefore

\[
\mathcal{E} + \lambda_1 \mathcal{G}(\mathbf{u}_o, \mathcal{E}, 0) = \lambda_1^2 \mathcal{G}(\mathbf{u}_o, \mathcal{G}(\mathbf{u}_g, \eta \mathbf{A}_g, 0), 0)
\]

\[
- \lambda_1 \mathcal{G}(\mathbf{u}_o - \mathbf{u}_g, \eta \mathbf{A}_g, 0) + \eta (\mathbf{A}_o - \mathbf{A}_g).
\]

Thus if \( \mathcal{G}(\mathbf{u}_g, \eta \mathbf{A}_g, 0) \) stays suitably bounded as \( \lambda_1 \to 0 \), then \( \mathcal{E} \) is \( \mathcal{O}(\lambda_1^2) \) when \( \mathbf{u}_o - \mathbf{u}_g \) is also \( \mathcal{O}(\lambda_1^2) \).

8 Some preliminary results

In this section, we present some preliminary results to amplify what has been proven so far. These are not complete in that they rely on information that is not known \textit{a priori}.

8.1 Identifying \( \mathbf{u}_x' \)

The derivatives \( \mathbf{u}_x' \) with respect to \( \lambda_1 \), as indicated in (5.19), can be determined as follows. We begin with the case of the grade-two model. Let

\[
\mathbf{w}_{\alpha_1} = \alpha_1^{-1}(\mathbf{u}_g - \mathbf{u}_s).
\]
Note that

$$u_G \cdot \nabla u_G - u_N \cdot \nabla u_N = \alpha_1 (w_{\alpha_1} \cdot \nabla u_G + u_N \cdot \nabla w_{\alpha_1}).$$

Therefore

$$-\eta \Delta w_{\alpha_1} + w_{\alpha_1} \cdot \nabla u_G + u_N \cdot \nabla w_{\alpha_1} + \nabla q_{\alpha_1} = \nabla \cdot G(u_G, A_1(u_G), 0), \quad (8.1)$$

where $q_{\alpha_1} = (p_o - p_N)/\alpha_1$. Although we know that $u_G \to u_N$ in $H^1(\mathcal{D})$ as $\alpha_1 \to 0$, it could well happen that $\nabla \cdot G(u_G, A_1(u_G), 0)$ does not go to $\nabla \cdot G(u_N, A_1(u_N), 0)$ as $\alpha_1 \to 0$. In particular, it may be that $\nabla \cdot G(u_N, A_1(u_N), 0)$ might not be in a suitable function space. Or it might be that $\nabla \cdot G(u_G, A_1(u_G), 0)$ does not go to anything reasonable at all as $\alpha_1 \to 0$.

But if suitable convergence does occur, the theorem below will prove that $u_G' = w$ where $w$ is a solution of

$$-\eta \Delta w + w \cdot \nabla u_N + u_N \cdot \nabla w + \nabla q = \nabla \cdot G(u_N, A_1(u_N), 0) \text{ in } \mathcal{D},$$

$$\nabla \cdot w = 0 \text{ in } \mathcal{D}, \quad w = 0 \text{ on } \partial \mathcal{D}. \quad (8.2)$$

**Theorem 8.1** Suppose that the assumptions of Lemma 2.1 hold and suppose that the data $f$ of this lemma are such that the solution $u_N$ of (3.14), satisfies

$$\|u_N\|_{H^1(\mathcal{D})} \leq \eta/2\sigma, \quad (8.3)$$

where $\sigma$ is defined in (3.15). Define the linear operator

$$Lz = -\eta \Delta z + z \cdot \nabla u_N + u_N \cdot \nabla z.$$ 

Then for each $f$ in $H^{-1}(\mathcal{D})^d$, the linear system

$$Lz + \nabla q = f \text{ in } \Omega, \quad \nabla \cdot z = 0 \text{ in } \Omega, \quad z = 0 \text{ on } \partial \mathcal{D}$$

has one and only one solution $z$ in $H^1_0(\mathcal{D})^d$ depending continuously on $f$, that is

$$\frac{\eta}{2} |z|_{H^1(\mathcal{D})} \leq \|f\|_{H^{-1}(\mathcal{D})}. \quad (8.4)$$

Moreover, let $u_G$ be a solution of (2.6) guaranteed by Lemma 2.1, then the limit

$$u_G' = w = \lim_{\alpha_1 \to 0} \alpha_1^{-1}(u_G - u_N) \quad (8.5)$$

is well defined and satisfies (8.2), provided only that

$$G(u_G, A_1(u_G), 0) \to G(u_N, A_1(u_N), 0) \text{ in } L^2(\mathcal{D}) \quad (8.6)$$

as $\alpha_1 \to 0$. 

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8.2 Proof of Theorem 8.1

As usual, problem (3.14) has at least one solution; furthermore, the assumptions of Lemma 2.1 imply that all solutions of (3.14) belong to $H^2(\mathcal{D})^d$. Therefore all terms of the operator $L$ are well-defined. In addition, the assumption (8.3) yields the invertibility of $L$ and the bound (8.4) for any data $f$ in $H^{-1}(\mathcal{D})^d$. It also gives the uniqueness of $u_N$. As $\nabla \cdot G(u_N, A_1(u_N), 0)$ belongs to $H^{-1}(\mathcal{D})^d$, these considerations also show that (8.2) defines uniquely $w$ in $H^1_0(\mathcal{D})^d$.

Now let $z = w_{\alpha_1} - w$. Then

\[-\eta \Delta z + w_{\alpha_1} \cdot \nabla u_G - w \cdot \nabla u_N + u_N \cdot \nabla z + \nabla (q_{\alpha_1} - q)\]

\[= \nabla \cdot \left( G(u_G, A_1(u_G), 0) - G(u_N, A_1(u_N), 0) \right).\]

But

\[w_{\alpha_1} \cdot \nabla u_G - w \cdot \nabla u_N = w_{\alpha_1} \cdot \nabla u_G - w \cdot \nabla u_G + w \cdot \nabla u_G - w \cdot \nabla u_N\]

\[= z \cdot \nabla u_N + (z + w) \cdot \nabla (u_G - u_N).\]

Therefore

\[Lz + (z + w) \cdot \nabla (u_G - u_N) + \nabla (q_{\alpha_1} - q) = \nabla \cdot \left( G(u_G, A_1(u_G), 0) - G(u_N, A_1(u_N), 0) \right),\]

(8.7)

hence,

\[Lz + \nabla (q_{\alpha_1} - q) = -(z + w) \cdot \nabla (u_G - u_N) + \nabla \cdot \left( G(u_G, A_1(u_G), 0) - G(u_N, A_1(u_N), 0) \right).\]

Note that the right-hand side of this equation belongs to $H^{-1}(\mathcal{D})^d$; therefore (8.4) implies

\[\frac{\eta}{2} |z|_{H^1(\mathcal{D})} \leq \| (z + w) \cdot \nabla (u_G - u_N) \|_{H^{-1}(\mathcal{D})}\]

\[+ \| \nabla \cdot \left( G(u_G, A_1(u_G), 0) - G(u_N, A_1(u_N), 0) \right) \|_{H^{-1}(\mathcal{D})}.\]

By assumption (8.6), the second term tends to zero as $\alpha_1 \to 0$. Regarding the first term, note that $z + w = w_{\alpha_1} = \frac{1}{\alpha_1} (u_G - u_N)$, so

\[\alpha_1 \| (z + w) \cdot \nabla (u_G - u_N) \|_{H^{-1}(\mathcal{D})} = \| (u_G - u_N) \cdot \nabla (u_G - u_N) \|_{H^{-1}(\mathcal{D})}\]

\[= \sup_{\phi \in H^1_0(\mathcal{D})^d} \left( \| \phi \|_{H^1(\mathcal{D})} \right)^2 \phi = \sup_{\phi \in H^1_0(\mathcal{D})^d} \left( \| \nabla \phi \|_{H^1(\mathcal{D})} \right)^2 \phi \leq \| u_G - u_N \|_{L^4(\mathcal{D})}^2.\]

Thus it remains to show that $\alpha_1^{-1} \| u_G - u_N \|_{L^4(\mathcal{D})}$ goes to zero as $\alpha_1 \to 0$. This follows from (5.18) by Sobolev's inequality. Thus we conclude that $z \to 0$ in $H^1(\mathcal{D})^d$ as $\alpha_1 \to 0$. QED

8.3 The Oldroyd case

We know that under, suitable smoothness conditions, we must have $u'_o = -\eta w$, where $w$ is given by (8.2). But it is interesting to derive such a limit to see what conditions are required for it to make sense. Let

\[w = \lambda_1^{-1} (u_G - u_N).\]
Note that
\[ \mathbf{u}_o \cdot \nabla \mathbf{u}_o - \mathbf{u}_N \cdot \nabla \mathbf{u}_N = \lambda_1 \left( \nabla \cdot \mathbf{u}_o + \mathbf{u}_N \cdot \nabla \mathbf{w} \right). \]
Therefore, analogous to (3.18), we have
\[ -\eta \Delta \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u}_o + \mathbf{u}_N \cdot \nabla \mathbf{w} + \nabla q = \nabla \cdot \mathbf{M}_o, \]
where \( q = (p_o - p_N)/\lambda_1 \) and, comparing to (3.12) with \( \mu_1 = 0 \), \( \mathbf{M}_o \) satisfies
\[ \mathbf{M}_o + \lambda_1 \mathcal{G}(\mathbf{u}_o, \mathbf{A}_1(\mathbf{u}_o), 0) = -\eta \mathcal{G}(\mathbf{u}_o, \mathbf{A}_1(\mathbf{u}_o), 0). \tag{8.8} \]
Thus we conclude that
\[ \mathbf{M}_o + \eta \mathcal{G}(\mathbf{u}_N, \mathbf{A}_1(\mathbf{u}_N), 0) \|_{L^2(\mathcal{D})} \leq \eta \| \mathcal{G}(\mathbf{u}_N, \mathbf{A}_1(\mathbf{u}_N), 0) - \mathcal{G}(\mathbf{u}_o, \mathbf{A}_1(\mathbf{u}_o), 0) \|_{L^2(\mathcal{D})} + |\lambda_1| \| \mathcal{G}(\mathbf{u}_o, \mathbf{M}_o, 0) \|_{L^2(\mathcal{D})}. \tag{8.9} \]
In view of the bound (2.5) and Lemma 3.1, we conclude that
\[ \mathbf{M}_o \to -\eta \mathcal{G}(\mathbf{u}_N, \mathbf{A}_1(\mathbf{u}_N), 0) \tag{8.10} \]
in \( L^2(\mathcal{D})^{d^2} \), provided that
\[ \mathcal{G}(\mathbf{u}_o, \mathbf{A}_1(\mathbf{u}_o), 0) \to \mathcal{G}(\mathbf{u}_N, \mathbf{A}_1(\mathbf{u}_N), 0) \text{ in } L^2(\mathcal{D})^{d^2} \text{ as } \lambda_1 \to 0. \]
Thus the convergence requirements for determining \( \mathbf{u}'_o \) are analogous to the ones for \( \mathbf{u}'_G \).

### 8.4 Understanding \( \mathcal{G} \)

The term \( \nabla \cdot \mathcal{G}(\mathbf{u}, \mathbf{A}_1(\mathbf{u}), 0) \) appears to be crucial in understanding the relationship between the Oldroyd models and the grade-two model. The highest-order term is clearly
\[ \nabla \cdot (\mathbf{u} \cdot \nabla (\mathbf{A}_1(\mathbf{u}))), \]
so it is useful to examine this term more closely. First of all
\[ \left( \mathbf{u} \cdot \nabla (\nabla \mathbf{u}' + \nabla \mathbf{u}') \right)_{ij} = \sum_k (u_{i,jk} + u_{j,ik}) u_k. \tag{8.11} \]
Therefore, using the fact that \( \mathbf{u} \) is divergence-free, we get
\[ \left( \nabla \cdot (\mathbf{u} \cdot \nabla (\nabla \mathbf{u}' + \nabla \mathbf{u}')) \right)_i = \sum_j \left( (u_{i,jk} + u_{j,ik}) u_k \right)_j \]
\[ = \sum_j (u_{i,j,k} + u_{j,i,k}) u_k + \sum_j (u_{i,j,k} + u_{j,i,k}) u_{k,j} \]
\[ = (\mathbf{u} \cdot \nabla (\Delta \mathbf{u}))_i + \sum_j u_k \sum_j (u_{j,j})_{ik} + \sum_{jk} (\mathbf{A}_1(\mathbf{u}))_{ij,k} u_{k,j} \tag{8.12} \]
\[ = (\mathbf{u} \cdot \nabla (\Delta \mathbf{u}))_i + \sum_j (\nabla \mathbf{A}_1(\mathbf{u}))_{ij,k} u_{k,j} \]
\[ = (\mathbf{u} \cdot \nabla (\Delta \mathbf{u}))_i + (\nabla \mathbf{A}_1(\mathbf{u}) \cdot \nabla \mathbf{u}'). \]
where the contraction denoted by the colon is defined accordingly. Thus we have found that
\[ \nabla \cdot (u \cdot \nabla (\nabla u + \nabla u')) = u \cdot \nabla (\Delta u) + \nabla A_1(u) : \nabla u'. \]

Therefore the leading term is the third-order derivative
\[ u \cdot \nabla (\Delta u). \]

That is,
\[ \nabla \cdot G(u, A_1(u), 0) = u \cdot \nabla (\Delta u) \]
plus terms involving only second-order, or lower-order, derivatives of \( u \). If \( u \) is a solution of the Navier-Stokes equations, then we can write
\[ \nabla \cdot G(u, A_1(u), 0) = u \cdot \nabla (\nabla p)/\eta \]
plus terms involving only second-order, or lower-order, derivatives of \( u \).

9 Further results

Extensions to the case \( \mu_1 \neq 0 (\alpha_1 + \alpha_2 \neq 0) \) appear to require a different approach to the Lipschitz continuity, since a straightforward approach to Theorem 6.1 appears to require that \( u^2 \in W^q_\alpha(D)^d \) for \( q > d \). Such a result is known in two dimensions [15], but the bounds on the norm degenerate like \( \alpha_1^{-1} \), and thus are not sufficient to yield a useful result. Indeed, although we know that \( u_G \rightarrow u_N \) in \( H^1(D) \) as \( \alpha \rightarrow 0 \), it is not known if \( \| u_G \|_{W^2_\alpha(D)} \) remains bounded in this limit, even in two dimensions (\( d = 2 \)).

Thus the case \( \mu_1 \neq 0 (\alpha_1 + \alpha_2 \neq 0) \) is completely open. Relaxing the smoothness conditions is more challenging and would likely require a substantially different technique of proof.

The grade-two model can be viewed as an Oldroyd model with \( \lambda_1 = \mu_1 = 0 \) and \( \lambda_2 = \alpha_1 \) and \( \mu_2 = \alpha_2 \) via (2.2). However, one normally assumes that \( 0 \leq \lambda_2 \leq \lambda_1 \) (and in fact \( \lambda_1 > 0 \)), and all theory for Oldroyd models is based on these assumptions. Thus this particular correspondence, although exact (not asymptotic), is only of use in providing a way of comprehending, via grade-two, Oldroyd models with such unusual coefficients.

10 Conclusions and questions

We have given a rigorous interpretation of the observation of Tanner [31]. However, our results suggest a requirement \( u_o \in H^3(D)^3 \) in order to get the full order of asymptotic similarity. This restriction is reasonable since it is related to the highest-order (third-order) term in the grade-two operator applied to the Oldroyd solution. This and other details of our proof raise some questions.

1. If the norm in (5.7), which involves third-order derivatives of the Oldroyd velocity \( u_o \), is not bounded as \( \lambda_1 \rightarrow 0 \), is the difference \( u_o - u_G \) still of second order in \( \lambda_1 \)? Otherwise said, is the condition \( u_o \in H^3(D)^3 \) necessary for second-order convergence?
2. Our proof of Theorem 7.4 requires that $f \in H^2(D)^d$. Is this necessary for the smoothness of the Oldroyd velocity $u_o$? How does less regularity for $f$ affect the asymptotic relationship between $u_o$ and $u_G$?

3. How does the behavior change if we take $\mu_1 \neq 0$? What can be proved if we take $\mu_1 \neq 0$?

These questions can be addressed computationally to some extent. We can limit the regularity of $u_o$ by picking the boundary geometry appropriately. Similarly, we can pick synthetic $f$ with appropriate smoothness.

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