# A SYMMETRIC ERROR ESTIMATE FOR GALERKIN APPROXIMATIONS OF TIME-DEPENDENT NAVIER-STOKES EQUATIONS IN TWO DIMENSIONS

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Abstract. A symmetric error estimate for Galerkin approximation of solutions of the Navier-Stokes equations in two space dimensions plus time is given. The finite dimensional function spaces are taken to be divergence free. The estimate is similar to known results for scalar parabolic equations. An application of the result is given for mixed method formulations.

## 1. Introduction

A symmetric error estimate for an numerical method is a statement that, in a certain norm, the error for the approximation produced by that method is best possible, up to a constant factor; i.e.,

$$\|error\| \leq C\|best\ approximation\ error\|.$$

For second order elliptic problems, the basic error estimate for the Galerkin approximations is a symmetric estimate in the $H^1$-norm. Some symmetric error estimates for parabolic equations can be found in [3, 4, 1, 5].

In this paper, we examine Galerkin approximations of solutions of the time-dependent two-dimensional Navier-Stokes equations on a bounded domain with zero boundary conditions. We consider the case in which the approximate solution is required to be divergence free. We prove stability for the Galerkin approximation on a finite time interval in a norm that is then used in the symmetric error estimate. In [2] an error estimate is given for the Stokes equation (see section 12.3) that is symmetric if one restricts to divergence-free spaces. In section 2, we introduce the boundary value problem and define some notion. In section 3, we define the Galerkin approximation and give a bound for it. In section 4, we state and prove the symmetric error estimate. In section 5, we show that the estimate applies to certain mixed method approximations of the same problem.

## 2. The Problem and notation

We are interested in approximating smooth solutions of the Navier-Stokes equations. Let $\Omega$ be a bounded domain in $\mathbb{R}^2$. Let for some $t_{final} > 0$, $J = (0, t_{final})$ and take $Q = \Omega \times J$. Suppose that $\overline{u} : \overline{Q} \to \mathbb{R}^2$, $p : \overline{Q} \to \mathbb{R}$ are smooth functions

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such that on $Q$

$$\overline{u}_t + \overline{u}\cdot\nabla\overline{u} + \frac{1}{\rho}\nabla p - K\triangle\overline{u} = F, \tag{2.1}$$

$$\nabla\cdot\overline{u} = 0, \tag{2.2}$$

where $\nabla$ is the spatial gradient operator and $\triangle$ is Laplacian and $F$ is a function in $L^2(Q)^2$. Here, $\rho$ and $K$ are positive constants, the density and kinematic viscosity, respectively. We suppose that $\overline{u}$ vanishes on the boundary of $\Omega$, $\partial\Omega$.

Let $\mathcal{X}$ be a norm space with norm $\|\cdot\|_{\mathcal{X}}$ and suppose that $f : J \to \mathcal{X}$ For sufficiently nice functions $f$ define for $1 \le p \le \infty$

$$\|f\|_{L^p(X)} = \|\|f(\cdot)\|_{\mathcal{X}}\|_{L^p(J)}.$$

Let $(\cdot,\cdot)_\Omega$ denote the $L^2(\Omega)$ inner product. We use this notation also for the inner product on $L^2(\Omega)^2$. For nonnegative integers $k$, we use the notation $\|f\|_k = \|f\|_{H^k(\Omega)}$, and we use the same notation for scalar or vector-valued functions. For $k = 0$, we use $\|\cdot\|$. For $\mathcal{M}$ a subspace of $H_0^1(\Omega)^2$ and $f \in L^2(\Omega)^2$, we define the following semi-norm:

$$\|f\|_{H^{-1}_{\mathcal{M}}} = \sup_{0\neq\sigma\in\mathcal{M}} \frac{(f,\sigma)_\Omega}{\|\nabla\sigma\|}.$$

A norm that plays a central role here is defined for (sufficiently nice) functions $f$ mapping $Q$ into $\mathbb{R}^2$

$$|||f|||^2 = \|f\|^2_{L^\infty(L^2(\Omega))} + \|f\|^2_{L^2(H^1_0(\Omega))} + \|f_t\|^2_{L^2(H^{-1}_{\mathcal{M}}(\Omega))}.$$

This norm depends on $\mathcal{M}$ of course.

## 3. Galerkin approximation and a Bound

Let

$$a(w,v) = K\int_\Omega \nabla w\nabla v,$$

$$c(w;s,v) = \int_\Omega (w\cdot\nabla s)\cdot v,$$

and note that for divergence-free functions $\phi \in H_0^1(\Omega)^2$,

$$(\overline{u}_t,\phi)_\Omega + c(\overline{u},\overline{u},\phi) + a(\overline{u},\phi) = (F,\phi)_\Omega, \quad t\in J\,, \tag{3.1}$$

Let $\mathcal{M}$ be a finite-dimensional subspace of the set of divergence-free functions in $H_0^1(\Omega)^2$. Define a function $\overline{U}$ mapping $\overline{J}$ into $\mathcal{M}$ by

$$(\overline{u}(0) - \overline{U}(0),\phi)_\Omega = 0, \quad \phi\in\mathcal{M}\,, \tag{3.2}$$

$$(\overline{U}_t,\phi)_\Omega + c(\overline{U},\overline{U},\phi) + a(\overline{U},\phi) = (F,\phi)_\Omega, \quad \phi\in\mathcal{M}, \quad t\in J\,, \tag{3.3}$$

where we identify $\overline{U}(t)$ with $\overline{U}(\cdot,\cdot,t)$. If $n$ is the dimension of $\mathcal{M}$ then (3.3) gives a system of $n$ ordinary differential equations that define the evolution of $\overline{U}$. The condition (3.2) defines $\overline{U}(0)$ as the $L^2(\Omega)$ projection of $\overline{u}(0)$ into $\mathcal{M}$. The existence of $\overline{U}$ on the entire interval follows if $\overline{U}$ is bounded on the interval where it is defined. We give such a bound here.

**Theorem 3.1.** *There is a constant $C$ that depends on $K$, $\|\overline{u}(0)\|$, $\|F\|_{L^2(H^{-1}_{\mathcal{M}})}$, and $\Omega$, such that*

$$|||U||| \le C.$$

*Proof.* If for each $t$ we replace $\phi$ in (3.2) by $\overline{U}(t)$ we get easily that

$$\frac{1}{2}\frac{d}{dt}\|\overline{U}\|^2+K\|\nabla\overline{U}\|^2 = (F,\overline{U}) \le \|F\|_{H^{-1}_{\mathcal{M}}}\|\nabla\overline{U}\| \le \frac{1}{2}K\|\nabla\overline{U}\|^2+\frac{1}{2K}\|F\|^2_{H^{-1}_{\mathcal{M}}}\,, \tag{3.4}$$

From this, it follows that for each $t \in J$

$$\|\overline{U}(t)\|^2 + K\int_0^t \|\nabla\overline{U}\|^2 \le \frac{1}{K}\int_0^t \|F\|^2_{H^{-1}_{\mathcal{M}}} + \|\overline{u}(0)\|^2\,, \tag{3.5}$$

Thus

$$\|\overline{U}\|^2_{L^\infty(L^2(\Omega))} + \|\overline{U}\|^2_{L^2(H^1(\Omega))} \le C$$

where $C$ depends on $K$, $\|\overline{u}(0)\|$, $\|F\|_{L^2(H^{-1}_{\mathcal{M}})}$, and $\Omega$.

Take $\phi \in \mathcal{M}$ such that $\|\nabla\phi\| = 1$ and $\|\overline{U}_t\|_{H^{-1}_{\mathcal{M}}} = (\overline{U}_t,\phi)_\Omega$. Then

$$\begin{aligned}\|\overline{U}_t\|_{H^{-1}_{\mathcal{M}}} &= -(\overline{U}\cdot\nabla\overline{U},\phi)_\Omega - K(\nabla\overline{U},\nabla\phi)_\Omega + (F,\phi)_\Omega\\ &= (\overline{U}\overline{U}^T,\nabla\phi)_\Omega - K(\nabla\overline{U},\nabla\phi)_\Omega + (F,\phi)_\Omega,\end{aligned}$$

where $\overline{U}$ is viewed as a column vector and $\overline{U}^T$ is its transpose. To be completely clear, we will write $\overline{U} = (U_1,U_2)^T$ and $\phi = (\phi_1,\phi_2)^T$, and note that

$$\begin{aligned}(\overline{U}\overline{U}^T,\nabla\phi)_\Omega &= \int_\Omega U_1^2\phi_{1,x} + U_1U_2\phi_{1,y} + U_1U_2\phi_{2,x} + U_2^2\phi_{2,y}\\ &\le \int_\Omega (U_1^4 + 2U_1^2U_2^2 + U_2^4)^{\frac{1}{2}}(\phi_{1,x}^2 + \phi_{1,y}^2 + \phi_{2,x}^2 + \phi_{2,y}^2)^{\frac{1}{2}}\\ &\le (\int_\Omega (U_1^2+U_2^2)^2)^{\frac{1}{2}}\|\nabla\phi\|\\ &\le (\int_\Omega |\overline{U}|^4)^{\frac{1}{2}} = \|\overline{U}\|^2_{L^4(\Omega)}.\end{aligned}$$

Since (see appendix)

$$\|\overline{U}\|^2_{L^4(\Omega)} \le C\|\overline{U}\|\|\nabla\overline{U}\|,$$

it follows that $\|\overline{U}_t\|_{H^{-1}_{\mathcal{M}}}$ is in $L^2(J)$ in time with a bound that depends only on $\|\overline{u}(0)\|$, $K$, and $\|F\|_{L^2(H^{-1}_{\mathcal{M}})}$. Hence $|||\overline{U}|||$ is bounded. □

## 4. Symmetric error estimate

Let $\overline{U}_1 = \overline{u}(\cdot,\cdot,t)$ be the solution of equations (2.1)-(2.2) and $\overline{U}_2 = \overline{U}_2(\cdot,\cdot,t)$ the Galerkin approximation. Define for any $B > 0$

$$S_B = \{(\varphi_1,\varphi_2) : \|\varphi_j\|_{L^\infty(Q)} \le B, \|\nabla\varphi_j\|_{L^\infty(Q)} \le B, j = 1,2\}.$$

We assume for a suitable $B$ that $\overline{U}_1 \in S_B$.

Take $\overline{U}_3$ to be any smooth function from $\overline{J}$ into $\mathcal{M}\cap S_B$. Our claim is that if $\overline{U}_3$ is close to $\overline{U}_1$ then $\overline{U}_2$ is also close to $\overline{U}_1$. That is, if the space $\mathcal{M}$ is sufficiently rich that we can approximate the solution of the Navier-Stokes equations well, then the Galerkin solution does a good job of approximating it. Here, we do not expect to use the minimum $B$ such that $\overline{U}_1 \in S_B$, so the constraint on $\overline{U}_3$ is rather weak. We are not assuming that the Galerkin solution $\overline{U}_2$ takes values in $S_B$. The generic constants that arise in the estimates below are allowed to depend on $B$. The use of constrained approximation for symmetric error estimates was introduced in [4] for nonlinear parabolic equations.

We adopt the addition notation

$$\overline{U}^{ij} = \overline{U}_i - \overline{U}_j. \tag{4.1}$$

**Theorem 4.1.** *There is a $C$ that depends on $B$, $K$, and $Q$, but not on $\mathcal{M}$, such that*

$$|||\overline{U}_1 - \overline{U}_2||| \le C|||\overline{U}_1 - \overline{U}_3||| \,. \tag{4.2}$$

*Proof.* Using (3.1) and (3.3) we see that for any $\phi \in \mathcal{M}$,

(4.3)

$$(\overline{U}^{23}_t, \phi)_\Omega + c(\overline{U}_2, \overline{U}^{23}, \phi) + a(\overline{U}^{23}, \phi) = (\overline{U}^{13}_t, \phi)_\Omega + c(\overline{U}_1, \overline{U}_1, \phi) - c(\overline{U}_2, \overline{U}_3, \phi) + a(\overline{U}^{13}, \phi).$$

Taking $\phi = \overline{U}^{23}$ and using the fact that $c(\overline{U}_2, \overline{U}^{23}, \overline{U}^{23}) = 0$ gives

$$\frac{1}{2}\frac{d}{dt}\|\overline{U}^{23}\|^2 + K\|\nabla\overline{U}^{23}\|^2 = (\overline{U}^{13}_t, \overline{U}^{23})_\Omega + (\overline{U}_1\cdot\nabla\overline{U}_1 - \overline{U}_2\cdot\nabla\overline{U}_3, \overline{U}^{23})_\Omega + K(\nabla\overline{U}^{13}, \nabla\overline{U}^{23})_\Omega.$$

Note that

$$\overline{U}_1 \cdot \nabla\overline{U}_1 - \overline{U}_2 \cdot \nabla\overline{U}_3 = \overline{U}_1 \cdot \nabla\overline{U}^{13} + \overline{U}^{13} \cdot \nabla\overline{U}_3 - \overline{U}^{23} \cdot \nabla\overline{U}_3.$$

Since $\overline{U}_1$,$\nabla\overline{U}_1$ and $\nabla\overline{U}_3$ are bounded by $B$,

$$\begin{aligned}|(\overline{U}_1 \cdot \nabla\overline{U}_1 - \overline{U}_2 \cdot \nabla\overline{U}_3, \overline{U}^{23})_\Omega| \le& \|\overline{U}_1\|_{L^\infty}\|\nabla\overline{U}^{13}\|\|\overline{U}^{23}\| + \|\overline{U}^{13}\|\|\nabla\overline{U}_3\|_{L^\infty}\|\overline{U}^{23}\| \\ &+ \|\overline{U}^{23}\|\|\nabla\overline{U}_3\|_{L^\infty}\|\overline{U}^{23}\| \\ \le& \epsilon\|\nabla\overline{U}^{23}\|^2 + C\|\overline{U}^{23}\|^2 + C\|\nabla\overline{U}^{13}\|^2\end{aligned}$$

where $C$ depends on $B$ and on $\epsilon$. The bounds

$$|(\overline{U}^{13}_t, \overline{U}^{23})_\Omega| \le C[\|\overline{U}^{13}_t\|^2_{H^{-1}_{\mathcal{M}}} + \|\overline{U}^{23}\|^2_1]$$

and

$$|(K\nabla\overline{U}^{13}, \nabla\overline{U}^{23})_\Omega| \le \epsilon\|\nabla\overline{U}^{23}\|^2 + C\|\nabla\overline{U}^{13}\|^2.$$

then imply (with $\epsilon = K/4$) that

$$\frac{d}{dt}\|\overline{U}^{23}\|^2 + K\|\nabla\overline{U}^{23}\|^2 \le C[\|\overline{U}^{13}_t\|^2_{H^{-1}_{\mathcal{M}}} + \|\overline{U}^{23}\|^2 + \|\nabla\overline{U}^{13}\|^2].$$

Now, Gronwall's inequality and the fact that $\|\overline{U}^{23}(0)\| \le \|\overline{U}^{13}(0)\|$ yield

$$\|\overline{U}^{23}\|^2_{L^\infty(L^2(\Omega))} + K\|\nabla\overline{U}^{23}\|^2_{L^2(Q)} \le C|||\overline{U}^{13}|||^2.$$

To complete the proof we need only bound the negative index semi-norm of $\overline{U}^{23}_t$. Take $\phi$ such that $\|\nabla\phi\| = 1$. From (4.3), we have

$$(\overline{U}^{23}_t, \phi)_\Omega = (\overline{U}^{13}_t, \phi)_\Omega + (\overline{U}_1\cdot\nabla\overline{U}_1 - \overline{U}_2\cdot\nabla\overline{U}_3, \phi)_\Omega + a(\overline{U}^{23}, \phi) - c(\overline{U}_2, \overline{U}_{23}, \phi) - a(\overline{U}^{13}, \phi).$$

This gives

$$\begin{aligned}|(\overline{U}^{23}_t, \phi)_\Omega| \le& |(\overline{U}^{13}_t, \phi)_\Omega| + |(\overline{U}_1 \cdot \nabla\overline{U}_1 - \overline{U}_2 \cdot \nabla\overline{U}_3, \phi)_\Omega| + |K(\nabla\overline{U}^{23}, \nabla\phi)_\Omega| \\ &+ |(\overline{U}_2 \cdot \nabla\overline{U}_{23}, \phi)_\Omega| + |K(\nabla\overline{U}^{13}, \nabla\phi)_\Omega|.\end{aligned}$$

Using

$$|(\overline{U}_1 \cdot \nabla\overline{U}_1 - \overline{U}_2 \cdot \nabla\overline{U}_3, \phi)_\Omega| \le C[\|\nabla\overline{U}^{13}\| + \|\overline{U}^{13}\| + \|\overline{U}^{23}\|]\|\nabla\phi\|$$

we get

$$|(\overline{U}^{23}_t, \phi)_\Omega| \le C\left(\|\nabla\overline{U}^{23}\| + \|\overline{U}^{13}_t\|_{H^{-1}_{\mathcal{M}}} + \|\nabla\overline{U}^{13}\| + \|\overline{U}^{13}\| + \|\overline{U}^{23}\|\right)\|\nabla\phi\| + |(\overline{U}_2\cdot\nabla\overline{U}^{23}, \phi)_\Omega|.$$

All of the terms in the big parenthesis are in $L^2(J)$. The last term on the right hand side can be bounded as follows:

$$|(\overline{U}_2 \cdot \nabla \overline{U}^{23}, \phi)_\Omega| \leq (\|\overline{U}^{23}\| \|\overline{U}^{23}\|_1 \|\overline{U}_2\| \|\overline{U}_2\|_1)^{1/2} \|\nabla \phi\|.$$

The last inequality follows a similar argument that was used in section 3. If $\mu(t) = (\|\overline{U}^{23}\| \|\overline{U}^{23}\|_1 \|\overline{U}_2\| \|\overline{U}_2\|_1)^{\frac{1}{2}}$, we have

$$\begin{aligned}
\int_J \mu^2 dt &\leq \|\overline{U}^{23}\|_{L^\infty(L^2)} \|\overline{U}_2\|_{L^\infty(L^2)} \int_J \|\overline{U}^{23}\|_1 \|\overline{U}_2\|_1 dt \\
&\leq \|\overline{U}^{23}\|_{L^\infty(L^2)} \|\overline{U}_2\|_{L^\infty(L^2)} \|\overline{U}^{23}\|_{L^2(H^1)} \|\overline{U}_2\|_{L^2(H^1)} \\
&\leq C[\|\overline{U}^{23}\|^2_{L^\infty(L^2)} + \|\overline{U}^{23}\|^2_{L^2(H^1)}].
\end{aligned}$$

Thus we have that $\|\overline{U}_t^{23}\|_{L^2(H^{-1}_{\mathcal{M}})} \leq C |\!|\!| \overline{U}^{13} |\!|\!|$. □

## 5. Application to mixed methods

Suppose that $V_h \subset H_0^1(\Omega)^2$ and $\Pi_h \subset L^2(\Omega)$ are finite dimensional spaces. Define

$$b(w, z) = -\int_\Omega (div\ w) z.$$

Then a mixed formulation approximation can be defined by requiring that for each $t \in J$

$$(\overline{U}_t, v)_\Omega + c(\overline{U}; \overline{U}, v) + a(\overline{U}, v) + b(v, P) = (F, v)_\Omega, \quad v \in V_h,$$

$$b(\overline{U}, q) = 0, \quad q \in \Pi_h,$$

where $\overline{U} : \overline{J} \to V_h$ and $P : \overline{J} \to \Pi_h$. The initial conditions $\overline{U}(0)$ and $P(0)$ are taken such that

$$(\overline{U}(0) - \overline{u}(0), v)_\Omega + b(v, P(0)) = 0, \quad v \in V_h,$$

$$b(U(0), q) = 0, \quad q \in \Pi_h.$$

Now suppose that $V_h$ and $\Pi_h$ have been chosen so that the set

$$\mathcal{M} = \{w \in V_h : b(w, q) = 0 \text{ for all } q \in \Pi_h\}$$

contains only divergence-free functions. Not all reasonable mixed method spaces will have this property, but examples do exist, for example, $\Pi_h = div\ V_h$. With this assumption, $\overline{U} : \overline{J} \to \mathcal{M}$, and, on $J$, $\overline{U}$ satisfies

$$(U_t, v)_\Omega + c(\overline{U}; \overline{U}, v) + a(\overline{U}, v) = (F, v)_\Omega, \quad v \in \mathcal{M}.$$

Also the initial value $\overline{U}(0)$ is in $\mathcal{M}$ and satisfies

$$(\overline{U}(0) - \overline{u}(0), v)_\Omega = 0, \quad v \in \mathcal{M}.$$

Hence the symmetric error estimate applies to $\overline{U}$.

## Appendix A. Bound on $L^4$ norm

**Lemma A.1.** *If $v \in H_o^1(\Omega)$ where $\Omega$ is a bounded domain in $R^2$ then*

$$\|v\|^4_{L^4(\Omega)} \leq \|v\|^2\|v_x\|\|v_y\| \leq \frac{1}{2}\|v\|^2\|\nabla v\|^2.$$

*Proof.* Without loss of generality, we can assume that $v \in C_o^\infty(\Omega)$ is extended by zero to $\mathbb{R}^2$. Suppose that $g$ has compact support on $\mathbb{R}$. Then it is elementary that

$$\max g^2 \leq \int |gg'|dx.$$

Now,

$$\begin{aligned}\int\int v^4 dxdy &\leq \int [\max_x v^2(x,y)\int v^2 dx]dy \\ &\leq (\max_y \int v^2(x,y)dx)(\int \max_x v^2(x,y)dy) \\ &\leq (\int \max_y v^2(x,y)dx)(\int \max_x v^2(x,y)dy).\end{aligned}$$

Hence

$$\begin{aligned}\|v\|^4_{L^4(\Omega)} \leq (\int\int |vv_y|dydx)(\int\int |vv_x|dxdy) \leq \|v\|^2\|v_y\|\|v_x\| \\ \leq \|v\|^2 \cdot \frac{1}{2}(\|v_y\|^2 + \|v_x\|^2) = \frac{1}{2}\|v\|^2\|\nabla v\|^2.\end{aligned}$$

□

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