A linear algebraic approach to representing and computing finite elements

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Abstract

We use standard ideas from numerical linear algebra plus orthogonal bases for polynomials to compute abstract finite elements. We express nodal bases in terms of orthogonal ones. This paradigm allows us to differentiate polynomials, evaluate local bilinear forms, and compute with affine equivalent elements. It applies not only to Lagrangian elements, but also to Hermite elements, $H(\mathrm{div})$ and $H(\mathrm{curl})$ elements, and elements subject to constraints.
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1 Introduction

Despite the general mathematical theory of finite elements covering many different kinds of spaces for many important problems in science and engineering, a similarly unified approach to computationally constructing finite element codes is yet to emerge. Many implementation strategies are *ad hoc*, and even most well-engineered codes fail to replicate the great generality of the mathematics. For example, codes are frequently written for a particular element type of a particular order with explicit formulae for the nodal basis functions. Though this is changing somewhat, a unified computational understanding of various finite elements is still lacking.

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In this paper, we present an abstract computational paradigm for finite elements that relies primarily on numerical linear algebra and the existence of orthonormal bases for widely-used function spaces. We focus on the element level, characterizing the basis for a finite element space and demonstrating how standard local bilinear forms may be evaluated via matrix computation. This approach underscores striking similarties between all elements and suggests unified strategy for implementing such "difficult" elements as Raviart-Thomas and Nedelec.

We begin with a basic observation of the equivalence between finite-dimensional function spaces and Euclidean space in Section 2. Namely, given an orthonormal basis for the function space, we may think of functions as sums of the components in these orthonormal modes. Vector space operations such as addition and scalar multiplication as well as inner products are transferred between the function space and Euclidean space. The existence of recurrence relations for orthogonal bases of polynomial spaces over standard element shapes makes this abstraction applicable in many situations.

However, not all function spaces used in finite element computations admit a "black box" orthogonal basis. We extend our results to many of these spaces by embedding them in larger ones for which such a basis is known, and transfer the computation into this larger space. This approach allows us to generate bases for polynomial spaces with edge constraints such as arise in p-adaptive finite elements as well as other scalar and vector polynomial spaces.

The conformity requirements of finite element methods typically do not admit direct usage of orthonormal bases. However, using Ciarlet's abstract definition of a finite element and its nodal basis, we compute nodal bases in terms of orthonormal ones and thence evaluate elementwise bilinear forms for finite elements via a change of basis. Section 3 details these computations.

We present three families of examples demonstrating the flexibility of our approach and its potential as the inner kernel of a general finite element code. We compute and plot basis functions for several different triangular elements. These include Lagrangian and Hermite elements with constrained degrees of freedom on edges and the Raviart-Thomas elements.

2 An equivalence between finite-dimensional function spaces and Euclidean space

2.1 Scalar-valued function spaces

Let $K \subset \mathbb{R}^d$ be a bounded domain with piecewise smooth boundary and $P \subset C\left(\bar{K}\right)$ a finite-dimensional space of functions from K to $\mathbb{R}$. Throughout, we denote by $(\cdot, \cdot)_K$ the L^2 inner product over K.

Let $\{\phi_i\}_{i=1}^{\dim P}$ be an L^2-orthonormal basis for P. In calculations, we assume that this basis is given by a "black-box" computer code that can evaluate the basis functions and their partial derivatives in the several coordinate directions at any point in $\bar{K}$.

Of course, any $p \in P$ may be written as

$$p = \sum_{i=1}^{P} (p, \phi_i)_K \, \phi_i. \tag{1}$$

Let $\mathcal{I} : P \to \mathbb{R}^{\dim P}$ be given by

$$(\mathcal{I}(p))_i = (p, \phi_i)_K \, , \quad p \in P, 1 \le i \le \dim P, \tag{2}$$

and we use the notation $\hat{p} \equiv \mathcal{I}(p)$ so that $\mathcal{I}(\cdot) = \hat{\cdot}$.

Proposition 2.1 *$\mathcal{I}$ is an isometric isomorphism between P and $\mathbb{R}^{\dim P}$.*

Proof. Clearly, $\mathcal{I}$ is bijective and preserves the vector space operations of addition and scalar multiplication. Also, for $p, q \in P$, the orthonormality of the basis functions allows us to write

$$\begin{aligned} (p, q)_K &= \left(\sum_{i=1}^{\dim P} \hat{p}_i \phi_i, \sum_{i=1}^{\dim P} \hat{q}_i \phi_i \right)_K \\ &= \sum_{i=1}^{\dim P} \hat{p}_i \hat{q}_i = \hat{p}^t \hat{q}. \end{aligned} \tag{3}$$

So, inner products are preserved under the mapping. $\square$

A computer program working with polynomials or other finite dimensional functions spaces then can add and subtract functions, scale them, and compute inner products simply by working with standard array operations. Differentiation and other linear mappings may also be applied by matrix operations once their action on the basis is determined.

Since we are supposing that we have a rule for evaluating the derivatives of the basis functions at particular points, we may express differentiation as bounded linear maps from P to P and describe corresponding matrix representations as follows.

Let $\{x_i\}_{i=1}^{\dim P} \subset \bar{K}$ be a set of unisolvent points. Recall that the points are unisolvent if for any $p \in P$, $p(x_i) = 0, 1 \le i \le \dim P$ only when $p(x) = 0 \forall x \in \bar{K}$. We let $\mathcal{E} : P \to \mathbb{R}^{\dim P}$ be the mapping evaluating functions at these points. That is,

$$(\mathcal{E}(p))_i = p(x_i) \, , \quad 1 \le i \le \dim P. \tag{4}$$

Similar to $\mathcal{I}$, we denote $\mathcal{E}(p) = \breve{p}$.

We introduce the van der Monde matrix $\mathsf{V} \in \mathbb{R}^{\dim P \times \dim P}$

$$\mathsf{V}_{i,j} = \phi_j(x_i). \tag{5}$$

The unisolvence of $\{x_i\}_{i=1}^{\dim P}$ ensures that V is nonsingular. However, some choices of unisolvent points can lead to ill-conditioned matrices, and other choices lead to good conditioning [14]. At any rate, V is a map from discrete modal coefficients to pointwise values. We may compose V with $\mathcal{I}$ to interpret it as a map from P to $\mathbb{R}^{\dim P}$.

Given a vector $\hat{p}$ of modal coefficients, we may compute its pointwise values by

$$\breve{p} = \mathsf{V}\hat{p}, \tag{6}$$

and if we know $\breve{p}$, we may find the vector of modal coefficients by solving the system.

Now, we let $\tilde{D}_\ell \in \mathbb{R}^{\dim P \times \dim P}$ be the matrix

$$\left(\tilde{D}_\ell\right)_{i,j} = \left(\frac{\partial \phi_j}{\partial x_\ell}\right)(x_i), \tag{7}$$

where $x_\ell, 1 \leq \ell \leq d$ is one of the coordinate directions. $\breve{q} = \left(\tilde{D}_\ell \circ \mathcal{I}\right)(p)$ then computes the partial derivative of p at the set of points. Note that $\mathcal{E}^{-1}\breve{q} = \frac{\partial p}{\partial x_\ell}$ in the sense of functions in P. We may compute its modal representation by solving

$$\mathsf{V}\hat{q} = \breve{q}. \tag{8}$$

Hence, the matrix $D_\ell = \mathsf{V}^{-1}\tilde{D}_\ell$ mapping modal coefficients of a function to modal coefficients of its derivative has entries

$$(D_\ell)_{i,j} = \left(\frac{\partial \phi_i}{\partial x_\ell}, \phi_j\right)_K. \tag{9}$$

Remark 2.1 *By using our representation for differentiation and computing inner products, we may compute inner products such as* $\left(\frac{\partial f}{\partial x_i}, g\right)_K$, $\left(\frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_j}\right)_K$ *as well. We use* $\mathcal{I}$ *to represent* f *and* g *as vectors of coefficients, then apply the matrices* D_i *as needed. For example,*

$$\left(\frac{\partial f}{\partial x_i}, g\right)_K = \left(D_i\hat{f}\right)^t \hat{g} \tag{10}$$

Thus, we may represent many standard operations over finite-dimensional function spaces when we have a code capable of evaluating an orthonormal basis for the space. In many spaces consisting of polynomials, such codes employ recurrence relations for the Jacobi polynomials. In many cases, we are dealing only with a subspace without such a nice basis. However, when $P \subset \bar{P}$ for some $\bar{P}$ with a computable basis, then we may still construct a basis for P. We focus on the abstract setting here and present examples later.

Let $\bar{P} \subset C\left(\bar{K}\right)$ be a finite-dimensional space of functions over K such that

- $\bar{P} \supset P$ and $\dim \bar{P} - \dim P = N$.
- $\{\bar{\phi}_i\}_{i=1}^{\dim \bar{P}}$ is an orthonormal basis for $\bar{P}$ for which we have a rule for evaluating and differentiating at particular points
- $\{\ell_i\}_{i=1}^{N}$ is a set of N linearly independent linear functionals over $\bar{P}$ with $\cap_{i=1}^{n} \mathrm{null}\,(\ell_i) = P$.

Remark 2.2 *The set of linear functionals above may be replaced with a set of linear transformations with domain $\bar{P}$, as long as the intersection of the corresponding null spaces is P.*

Remark 2.3 *A similar approach is used by Dupont and Scott [7] in the context of approximation theory, where they use the kernel of differential operators to characterize spaces of polynomials. We use these "annihilators" in a computational rather than analytic paradigm.*

Remark 2.4 *In some cases, it may be beneficial to write down some basis that is not orthogonal and then orthogonalize it numerically rather than characterize the space by annihilators. Such is probably the case with the even-degree primal nonconforming elements introduced by Raviart and Thomas [11], which add a single function to polynomials of complete degree. We do not dwell on this situation here.*

We may use this arrangement to compute an orthonormal basis for P. First, we recall the following theorem from linear algebra:

Theorem 1 (Singular Value Decomposition) *Let $A \in \mathbb{R}^{m\times n}$. Then $\exists U_A \in \mathbb{R}^{m\times m}, \Sigma_A \in \mathbb{R}^{m\times n}$, and $V_A \in \mathbb{R}^{n\times n}$ such that*

1. $A = U_A \Sigma_A V_A^t$

2. U_A, V_A *are orthogonal matrices*

3. Σ_A *is diagonal*

Moreover, if $r = rank(A)$, then the first r columns of U_A span the range of A and the last r columns of V_A span the null space of A.

We may compute an orthonormal basis for P as follows:

Proposition 2.2 *Now, let $L \in \mathbb{R}^{N,\dim P}$ be the matrix given by*

$$L_{i,j} = \ell_i\left(\bar{\phi}_j\right) \tag{11}$$

and $U_L \Sigma_L V_L^t$ be its singular value decomposition. Moreover, let $\mathcal{V} \in \mathbb{R}^{\dim \bar{P} \times \dim P}$ be the matrix given by

$$\mathcal{V}_{i,j} = \left(V_L\right)_{i,j+N}. \tag{12}$$

Then the set

$$\left\{ \phi_j = \sum_{i=1}^{\dim \bar{P}} \mathcal{V}_{i,j} \bar{\phi}_i \right\}_{j=1}^{\dim P} \tag{13}$$

forms an orthonormal basis for P.

Proof. There are two things to demonstrate. First, we must show that this set is indeed in P. Second, demonstrating its orthonormality establishes its linear independence and hence by dimensionality that it spans P.

First, we establish that $\phi_j \in P$, $1 \leq j \leq \dim P$. This is equivalent to the condition that $\ell_k\left(\phi_j\right) = 0$ for $1 \leq k \leq N$. So then,

$$\begin{aligned} \ell_k\left(\phi_j\right) &= \ell_k\left(\sum_{i=1}^{\dim \bar{P}} \mathcal{V}_{i,j} \bar{\phi}_i\right) \\ &= \sum_{i=1}^{\dim \bar{P}} \mathcal{V}_{i,j} \ell_k\left(\bar{\phi}_i\right) \\ &= \sum_{i=1}^{\dim \bar{P}} L_{k,i} \mathcal{V}_{i,j} \\ &= 0, \end{aligned} \tag{14}$$

since the columns of $\mathcal{V}$ are in the null space of L. Hence, $\left\{\phi_j\right\}_{j=1}^{\dim P} \subset P$.

We rely on the isometry between $\bar{P}$ and $\mathbb{R}^{\dim \bar{P}}$ to establish orthonormality. Let the vectors $\left\{v_j\right\}_{j=1}^{\dim P} \subset \mathbb{R}^{\dim \bar{P}}$ be the columns of $\mathcal{V}$. That is,

$$\left(v_j\right)_i = \mathcal{V}_{i,j}, \quad 1 \leq i \leq \dim \bar{P}, 1 \leq j \leq \dim P. \tag{15}$$

Then,

$$\begin{aligned} \left(\phi_i, \phi_j\right)_K &= \left(\sum_{k=1}^{\dim \bar{P}} \mathcal{V}_{k,i} \bar{\phi}_k, \sum_{k=1}^{\dim \bar{P}} \mathcal{V}_{k,j} \bar{\phi}_k\right)_K \\ &= \left(v_i\right)^t \left(v_j\right) \\ &= \delta_{i,j}, \end{aligned} \tag{16}$$

owing to the fact that $\mathcal{V}$ is an orthogonal matrix. ∎

Remark 2.5 *The matrix $\mathcal{V}$ embeds P in $\bar{P}$. That is, given a vector $\hat{p}$ consisting of modal coefficients $(p, \phi_i)_K$, the vector $\mathcal{V}\hat{p} \equiv \hat{\bar{p}}$ consists of modal coefficients $\left(p, \bar{\phi}_i\right)_K$*

Remark 2.6 *The matrix $\mathcal{V}^t$ projects $\bar{P}$ onto P.*

We may use this representation to apply linear functionals and linear transformations on P.

Let $\ell : \bar{P} \to \mathbb{R}$ be a bounded linear functional on $\bar{P}$. We may represent ℓ as a vector in $\mathbb{R}^{\dim \bar{P}}$ by

$$\left(\bar{\ell}'\right)_i = \ell\left(\bar{\phi}_i\right), \quad 1 \leq i \leq \dim \bar{P}, \tag{17}$$

for then,

$$\begin{aligned} \ell(p) &= \ell\left(\sum_{i=1}^{\dim \bar{P}} \hat{\bar{p}}_i \bar{\phi}_i\right) \\ &= \sum_{i=1}^{\dim \bar{P}} \hat{\bar{p}}_i \ell\left(\bar{\phi}_i\right) \\ &= \left(\bar{\ell}'\right)^t \hat{\bar{p}} \end{aligned} \tag{18}$$

From here, we may interpret $\ell \in \mathbb{R}^{\dim P}$ by the vector

$$\ell' = \bar{\ell}'\mathcal{V}, \tag{19}$$

and it can be readily seen that

$$(\ell')_i = \ell\left(\phi_i\right). \tag{20}$$

Proposition 2.3 *Let $\mathbb{T} : \bar{P} \to \bar{P}$ be a bounded linear map with $\mathbb{T} : P \to P$. Let the matrix representation of $\mathbb{T}$ acting on $\bar{P}$ be given:*

$$\bar{T}_{i,j} = \left(\mathbb{T}\bar{\phi}_i, \bar{\phi}_j\right)_K, \tag{21}$$

The corresponding matrix for $\mathbb{T}$ acting only on P,

$$T_{i,j} = \left(\mathbb{T}\phi_i, \phi_j\right)_K. \tag{22}$$

is given by

$$T = \left(\bar{T}\mathcal{V}\right)^t \mathcal{V} = \mathcal{V}^t \bar{T}^t \mathcal{V} \tag{23}$$

Proof. Directly expand ϕ_i, ϕ_j in (22) using (13). ∎

2.2 Vector-valued function spaces

Now consider some space $\mathbf{P} = \Pi_{k=1}^{d} P$ consisting of functions mapping K into $\mathbb{R}^d$, where $P \subset C\left(\bar{K}\right)$ is a finite-dimensional function space. We denote the k^{th} component of some $p \in \mathbf{P}$ by $p^k \in P$.

As before, let $\{\phi_i\}_{i=1}^{\dim P}$ be an orthonormal basis for P. Let $\operatorname{div}(a,b), \operatorname{mod}(a,b)$ denote the standard number-theoretic integer division and remainder operations, respectively. For $1 \leq i \leq \dim \mathbf{P} = d \dim P$, we let

$$\kappa_i = (i-1) \operatorname{div}(i, \dim P) + 1, \tag{24}$$

$$\omega_i = (i-1) \operatorname{mod}(i, \dim P) + 1, \tag{25}$$

and $e^{(k)}$ be the canonical basis vector in $\mathbb{R}^d$. Then,

$$\left\{\zeta_i = \phi_{\kappa_i} e^{(\omega_i)}\right\}_{i=1}^{\dim \mathbf{P}} \tag{26}$$

forms an orthonormal basis for $\mathbf{P}$. This basis is, of course, simply taking the first $\dim P$ elements to consist of functions with the basis for P in the first component and zero in the rest, the second $\dim P$ functions to have only a second component, and so on. We maintain the notation $\mathcal{I}(p) = \hat{p}$ to denote the mapping from a function to its expansion coefficients.

Now, for $q \in \mathbf{P}$ and $\hat{q}$ its vector of modal coefficients, it is useful to extract those modes with only support in the k^{th} component. To this, we introduce $\Xi^k : \mathbb{R}^{\dim \mathbf{P}} \rightarrow \mathbb{R}^{\dim P}$ by

$$\left(\Xi^k \hat{p}\right)_i = \hat{p}_{(k-1)\dim P + i}, \tag{27}$$

and let $\hat{p}^k = \Xi^k \hat{p}$

With this formalism, we may compute $\left(L^2\right)^d$ inner products by simply computing the inner products of each component and summing. Let $p, q \in \mathbf{P}$ and then

$$(p,q)_K = \sum_{k=1}^{d} \left(p^k, q^k\right)_K = \sum_{k=1}^{d} \left(\Xi^k \hat{p}\right)^t \left(\Xi^k \hat{q}\right) \tag{28}$$

We may also use the derivative operations developed previously to implement vector gradients, divergences, and curls.

3 Finite elements

Having seen that finite-dimensional function spaces may be manipulated through linear algebraic operations, we now apply this idea in the context of finite element function spaces. We recall the definition of Ciarlet [5]

Definition 1 *A* finite element *is a triple* (K, P, N)*, where*

i $K \subset \mathbb{R}^n$ *is a domain with piecewise smooth boundary,*

ii P *is a finite-dimensional space of functions over* K*, and*

iii $N = \{n_i\}_{i=1}^{\dim P}$ *is a basis for* P'.

Of importance in finite element computations is the *nodal basis*, which allows us to enforce continuity between elements:

Definition 2 *Let* (K, P, N) *be a finite element. The basis* $\{\psi_i\}_{i=1}^{\dim P}$ *for* P *dual to* N *(i.e.* $n_i(\psi_j) = \delta_{i,j}$*) is called the* nodal basis *for* P.

We many interpolate functions using the nodal basis.

Definition 3 *Let* (K, P, N) *be a finite element and let* $f \in C(\bar{K})$*. Then, the* nodal interpolant *of* f *is*

$$\mathrm{I}(f) = \sum_{i=1}^{\dim P} n_i(f)\,\psi_i \tag{29}$$

If $f \in P$, the $\mathrm{I}(f) = f$

We typically work with the nodal basis in finite element calculations to enforce appropriate measures of continuity between adjacent domains. However, the members of a nodal basis can be difficult to evaluate explicitly. Many finite element codes rely on an exact formula worked out for each basis function. If the code allows several different orders of approximation, each must be put into the code.

We present an approach that allows much more generality. We may evaluate the nodal basis in terms of an orthonormal basis. As before, we let $\{\phi_i\}_{i=1}^{\dim P}$ be an orthonormal basis for P. We introduce the generalized van der Monde matrix V with

$$V_{i,j} = n_i(\phi_j) \tag{30}$$

Each nodal basis functions $\{\psi_i\}_{i=1}^{\dim P}$ may be expanded in terms of $\{\phi_i\}_{i=1}^{\dim P}$

$$\psi_k = \sum_{j=1}^{\dim P} \alpha_j^{(k)} \phi_j, \quad 1 \le k \le \dim P \tag{31}$$

Proposition 3.1 *The coefficients* $\left\{\alpha_j^{(k)}\right\}_{j,k=1}^{\dim P}$ *of the expansions of the nodal basis functions in terms of the orthonormal components are given by*

$$\alpha_j^{(k)} = V_{j,k}^{-1}. \tag{32}$$

Proof. Applying n_i to equation (31) for $1 \leq i \leq \dim P$ and using the fact that the basis is nodal gives rise to a system of $\dim P$ linear equations

$$\sum_{j=1}^{\dim P} \alpha_j^{(k)} n_i \left(\phi_j\right) = \sum_{j=1}^{\dim P} V_{i,j} \alpha_j^{(k)} = \delta_{i,k}. \tag{33}$$

Hence, the coefficients $\left\{\alpha_j^{(k)}\right\}_{j=1}^{\dim P}$ are computed by solving the system

$$V\alpha^{(k)} = e^{(k)}, \tag{34}$$

where $\alpha^{(k)}$ is the vector $\left(\alpha^{(k)}\right)_j = \alpha_j^{(k)}$ and $e^{(k)}$ is the canonical basis vector in $\mathbb{R}^{\dim P}$. So then, we may solve for all the coefficients simultaneously by introducing multiple right hand sides. Let $A_{i,j} = \alpha_i^{(j)}$ be the matrix of coefficients. Then we have

$$VA = I, \tag{35}$$

□

Remark 3.1 *In certain situations, an actual matrix-vector product is not necessary to compute the modal representation, as faster algorithms such as the FFT are applicable.*

Our representation of the nodal basis in terms of orthonormal functions enables us to apply the methods of Section 2 for computing operations by a change of basis We let $\mathcal{N} : P \rightarrow \mathbb{R}^{\dim P}$ be the mapping of an element of P to the vector of its nodal coeffictions. That is,

$$\left(\mathcal{N}\left(f\right)\right)_i = \left(\check{f}\right)_i = n_i\left(f\right) \tag{36}$$

3.1 Scalar-valued finite elements

Here, we bring together the general ideas of finite elements with the discussion in 2.1. Let $f, g \in P$ and suppose we have $\check{f}, \check{g}$ the vectors of nodal coefficients.

We compute the vectors of modal coefficients

$$\hat{f} = V^{-1}\check{f}, \quad \hat{g} = V^{-1}\check{g}, \tag{37}$$

and hence,

$$(f,g)_K = \hat{f}^t \hat{g} = \left(V^{-1}\check{f}\right)^t \left(V^{-1}\check{g}\right) = \check{f}^t V^{-t} V^{-1} \check{g}. \tag{38}$$

Note that $\check{\psi}_i = e^{(i)}$. We may compute the mass matrix $M_{i,j} = (\psi_i, \psi_j)_K$ by

$$M = V^{-t} V^{-1}, \tag{39}$$

or we may apply it to a vector by

$$Mu = V^{-t} V^{-1} u. \tag{40}$$

We may also change basis and use the operator D_ℓ introduced previously to compute inner products involving derivatives.

$$\left(\frac{\partial f}{\partial x_\ell}, g\right)_K = \left(D_\ell \hat{f}\right)^t (\hat{g}) = \left(D_\ell V^{-1} \check{f}\right)^t \left(V^{-1}\check{g}\right), \tag{41}$$

$$\left(\frac{\partial f}{\partial x_\imath}, \frac{\partial g}{\partial x_\jmath}\right)_K = \left(D_\imath V^{-1} \check{f}\right)^t \left(D_\jmath V^{-1} \check{g}\right) \tag{42}$$

Note that if we are in d space dimensions, then the local stiffness matrix over K is $S_{i,j} = (\nabla \psi_i, \nabla \psi_j)_K$ may be computed by

$$S = \sum_{k=1}^{d} \left(D_k V^{-1}\right)^t \left(D_k V^{-1}\right) \tag{43}$$

3.2 Vector-valued finite elements

As before, we may characterize the nodal basis functions in terms of the orthonormal basis. The Brezzi-Douglas-Marini elements [4] are treated along these lines in [10]. To supplement the above discussion, we discuss applying standard differential operators. We will map from nodal coefficients to modal coefficients of the derivatives.

Given $q \in \mathbf{P}$, and $\check{q}$ its nodal expansion, we may compute its divergence in P as follows. First, we convert to the orthonormal modes. From this, we may extract the modes corresponding to each component, apply the proper derivative, and sum over components. This is expressed as

$$\mathcal{I}(\nabla \cdot q) = \left(\sum_{k=1}^{d} D_k \Xi^k\right) V^{-1} \check{q}. \tag{44}$$

In other situations, such as mixed methods, we have two function spaces, $\mathbf{P} = \Pi_{i=1}^{d} P$ and Q such that the divergence maps $\mathbf{P}$ onto Q. In this case, we may proceed in one of two ways.

First, we may use the standard derivative operators from P to P and then project the result onto $Q \subset P$. On the other hand, we may redefine the operators $\tilde{D}_\ell$ to map from modes of P onto only enough points to span Q. With either approach, we may develop an appropriate matrix representation of differentiation. We may compute similar matrix representations of vector gradients and curls.

A further generalization is needed when the vector-valued function space is not a simple Cartesian product of a single space. This occurs in some $H(\mathrm{div})$ elements such as Raviart-Thomas and Brezzi-Douglas-Fortin-Marini [12, 3]. In this case, the spaces neither consist of complete polynomial degrees. However, the vector spaces naturally lie in a space $\bar{\mathbf{P}}$ consisting of the Cartesian product of a single space of polynomials of complete degree.

3.3 Some extensions

For many finite elements, we do not have a convenient orthonormal basis. Sometimes, we can only define a basis through a change of variables from some canonical shape. Also, for some elements, the function space itself does not permit a simple orthonormal basis, such as when polynomial degrees on the boundary are constrained to be less than that of the interior. Here, we show how the formalism in this paper can be extended in two ways. First, we demonstrate how we may compute modal bases through *affine equivalence*. Then, we proceed to extend the discussion of constrained spaces in the previous section to the case of finite elements.

3.3.1 Affine equivalence

For some finite element shapes such as triangles, computations can be performed on a "reference element" and then transferred to the "physical element" by means of an affine mapping We follow, for example, Brenner and Scott [1] in defining affine equivalence.

Definition 4 *Let (K, P, N) be a finite element and let $F(x) = \mathrm{A}x + \mathrm{b}$ with A nonsingular be an affine map. Then the finite element $(\acute{K}, \acute{P}, \acute{N})$ is* affine equivalent *to (K, P, N) if*

i $F(K) = \acute{K}$,

ii $F^* \acute{P} = P$,

iii $F_* N = \acute{N}$,

where $F^*(\acute{f}) = \acute{f} \circ F$ and $(F_* N)(\acute{f}) = N(F^*(\acute{f}))$.

We may use this formalism to compute transfer orthonormal bases between equivalent elements.

Proposition 3.2 *Let $(\acute{K}, \acute{P}, \acute{N})$ and (K, P, N) be affine equivalent finite elements with $F : K \to \hat{K}$ the associated affine map with Jacobian determinant J. Suppose that $\left\{\acute{\phi}_i\right\}_{i=1}^{\dim \acute{P}}$ is an orthonormal basis for $\acute{P}$. Then the set $\{\phi_i\}_{i=1}^{\dim P}$, where $\phi_i = \frac{1}{\sqrt{J}} F^* \left(\acute{\phi}_i\right)$, forms an orthonormal basis for P.*

Proof. Use change of variables from calculus. ∎

Evaluating a single van der Monde matrix is sufficient for an entire family of affine equivalent elements.

Proposition 3.3 *Let $(\acute{K}, \acute{P}, \acute{N})$ and (K, P, N) be affine equivalent finite elements with $F : K \to \hat{K}$ the associated affine map with Jacobian determinant J. Let $\acute{V}_{i,j} = \acute{n}_i \left(\acute{\phi}_j\right)$ and $V_{i,j} = n_i (\phi_j)$ be the respective van der Monde matrices. Then*

$$V = \frac{1}{\sqrt{J}} \acute{V} \tag{45}$$

Proof. First, we compute V:

$$V_{i,j} = n_i (\phi_j) = n_i \left(\frac{1}{\sqrt{J}} F^* \left(\acute{\phi}_j\right)\right) = \frac{1}{\sqrt{J}} n_i \left(F^* \left(\acute{\phi}_i\right)\right) \tag{46}$$

Then, we compute $\acute{V}$:

$$\acute{V}_{i,j} = \acute{n}_i \left(\acute{\phi}_j\right) = (F_* (n_i)) \left(\acute{\phi}_j\right) = n_i \left(F^* \left(\acute{\phi}_j\right)\right) \tag{47}$$

∎

We may transfer the derivative matrices from a reference element to any other equivalent element as follows:

Proposition 3.4 *Let $(\acute{K}, \acute{P}, \acute{N})$ and (K, P, N) be affine equivalent finite elements with $F = Ax+b$ the associated affine map with Jacobian determinant J. Let $\left\{\acute{\phi}_i\right\}_{i=1}^{\dim \acute{P}}$ be an orthonormal basis for $\acute{P}$, and define an orthonormal basis for P as in Proposition 3.2 Let $\acute{D}_\ell$ be the derivative matrix in the ℓ coordinate direction on $\acute{K}$ as in (9). Then*

$$D_i = \sum_{j=1}^{\dim P} \left(A^{-1}\right)_{i,j} \acute{D}_j \tag{48}$$

Proof. Use the chain rule and the calculations leading up to (9). ∎

3.3.2 Constrained spaces

The van der Monde matrix plays an important role in finite element operations. We now consider computing the van der Monde matrix for constrained spaces. Let (K, P, N) be a finite elements with P embedded in some larger space $\bar{P}$ for which we may compute some orthonormal basis as before. We let the matrix $\mathcal{V}$ be computed as the matrix whose columns span P just as in (12). We may compute the van der Monde matrix for the finite element by a matrix multiplication:

Proposition 3.5 *Let $\bar{V}$ be the matrix with*

$$\bar{V}_{i,j} = n_i\left(\bar{\phi}_j\right), \tag{49}$$

where $\{n_i\}_{i=1}^{\dim P}$ are the nodes for the finite element and $\left\{\bar{\phi}_i\right\}_{i=1}^{\dim \bar{P}}$ is the given orthonormal basis for $\bar{P}$. Then, the van der Monde matrix is

$$V = \bar{V}\mathcal{V}. \tag{50}$$

Proof. Express the modal basis for P by (13) and use the linearity of the nodes. □

4 Examples

Hesthaven and Warburton [9] have used the duality between nodal and modal representations in the context of Lagrangian elements. Here, we focus on other, more general elements. In each case, we let K to be the triangle consisting of vertices $x_1 = (-1, 1), x_2 = (-1, -1), x_3 = (1, -1)$. Dubiner [6] has demonstrated an orthonormal basis for polynomials of total degree on this domain.

4.1 Lagrangian elements

As a simple example, we let P be the set of polynomials of degree 2 over K. The Dubiner basis for P_2 is the set of polynomials

$$\begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}\,(1+2\,x+y)}{2} \\ \frac{1+3\,y}{2} \\ \frac{\sqrt{\frac{15}{2}}\left(1+6\,x^2+4\,y+y^2+6\,x\,(1+y)\right)}{4} \\ \frac{3\,(1+2\,x+y)\,(3+5\,y)}{4\,\sqrt{2}} \\ \sqrt{\frac{3}{2}}\,\left(-\left(\frac{1}{2}\right)+y+\frac{5\,y^2}{2}\right) \end{array} \tag{51}$$

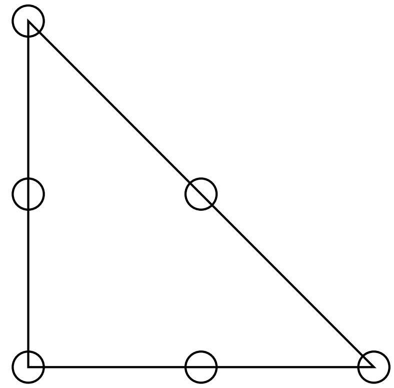

Figure 1: Node locations for quadratic Lagrange element

In this case, our set of nodes is pointwise evaluation at the vertices and edge midpoints, as seen in Figure 4.1. We may compute the van der Monde matrix either directly from the formulae in (51) or from recurrence relations.

In order to evaluate the nodal basis functions at a set of points $\{x_i\}_{i=1}^{n}$, we evaluate each Dubiner function at each of these points, resulting in the matrix V with

$$\mathsf{V}_{i,j} = \phi_j\left(x_i\right). \tag{52}$$

Then, we compute

$$Y = \mathsf{V}V^{-1}. \tag{53}$$

The j^{th} column of Y contains the values of the j^{th} nodal basis function at each point. Two of these functions are plotted in Figures 4.1 and 4.1

While this approach may seem tedious for low-degree elements like quadratics, it extends to any order polynomials. As a simple example of constrained spaces, we also consider the case in which we use quadratic polynomials constrained to be linear along the bottom edge γ_1. Implementing such a function space is necessary in so-called $p-$ and $hp-$ finite element methods. The five nodes are evaluation at the vertices and the midpoints of the other two edges. These nodes are depicted in Figure 4.1.

We let $\bar{P} = P_2\left(K\right)$ and take the orthonormal basis $\left\{\bar{\phi}_i\right\}_{i=1}^{6}$ to be the Dubiner functions above. The space P is

$$P = \left\{p \in \bar{P} : p|_{\gamma_1} \in P_1\right\} \tag{54}$$

We may formalize the constraint on $\bar{P}$ as

$$\ell\left(p\right) = \int_{\gamma_1} p\mu_2 \mathrm{ds} = 0, \tag{55}$$

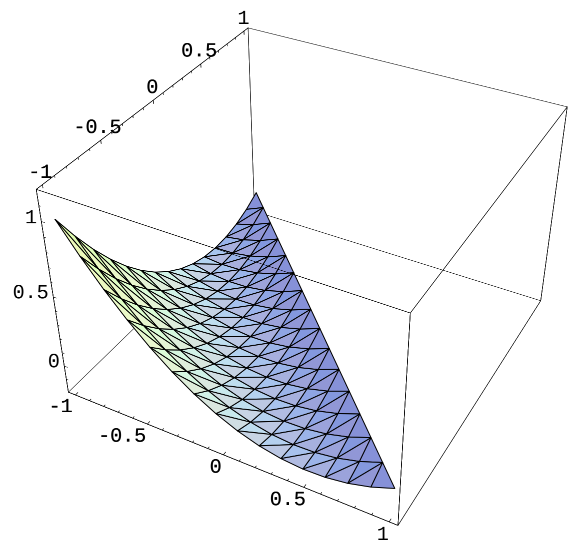

Figure 2: A nodal basis function for Lagrangian quadratics

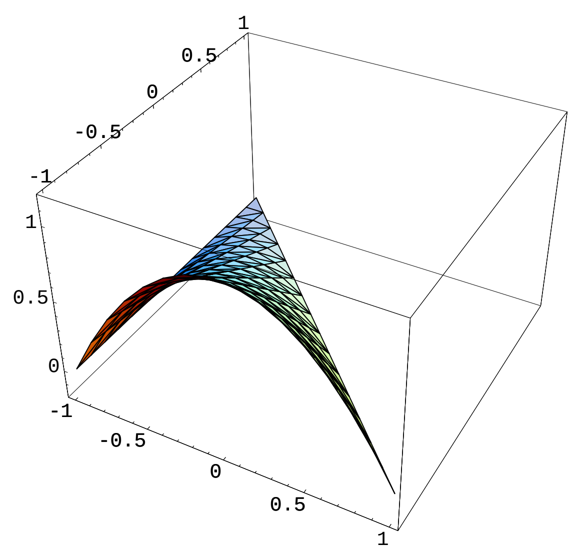

Figure 3: Another nodal basis function for Lagrangian quadratics

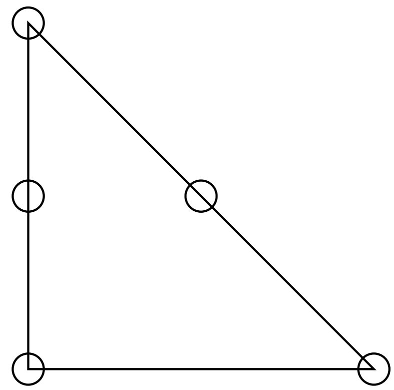

Figure 4: Node locations for constrained quadratic Lagrange element

where μ_2 is the quadratic Legendre polynomial. By evaluating this linear functional on each Dubiner function, we obtain that

$$L = \begin{pmatrix} 0 & 0 & 0 & \sqrt{\frac{6}{5}} & 0 & 0 \end{pmatrix} \tag{56}$$

Now, we may tabulate the nodal basis functions at a lattice of points by letting the new matrix $\bar{\mathsf{V}}$ be V from above. We compute a new matrix $\tilde{\mathsf{V}}$ from

$$\tilde{\mathsf{V}} = \tilde{\bar{V}}\mathcal{V}, \tag{57}$$

which tabulates the orthonormal basis for the constrained space. Then, we may evaluate the nodal functions by

$$\tilde{Y} = \tilde{\mathsf{X}} V^{-1} \tag{58}$$

Figures 4.1 shows one of these constrained nodal basis functions.

4.2 Hermite elements

The Hermite elements on triangles are C^0 and have continuous derivatives at the vertices. They may be used as a C^1 nonconforming element in some situations and also have some advantages over Lagrangian elements in Stokes calculations in that they simply handling of singular vertices [13].

Hermite elements use the same function space as Lagrangian ones, but employ a different set of nodes. The nodes for the Hermite elements of order $k \geq 3$ are

- pointwise evaluation at the vertices

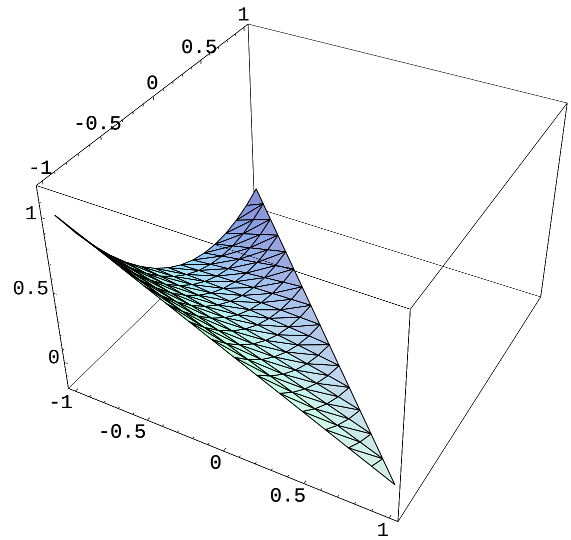

Figure 5: A nodal basis function for constrained Lagrangian quadratics

- evaluation of the partial derivatives at the vertices
- pointwise evaluation at $\dim P_{k-3}$ points at the interior of the triangle
- pointwise evaluation at $k-3$ points along each edge of the triangle

While building the van der Monde matrix is slightly more involved than previously, exactly the same process enables us to represent the nodal functions as linear combinations of the orthonormal modes. By evaluating the modes, we may evaluate the functions and plot them. Three of the functions, corresponding to unit height at a vertex and unit directional derivatives at the same vertex are seen in Figures 4.2, 4.2, and 4.2.

We may also create nodal Hermite elements constrained on edges, as would occur in finite element codes with variable polynomial degree. Figure 4.2 shows a sample nodal basis function of the space of Hermite quintics constrained to be cubic on one edge and quartic on another.

4.3 Raviart-Thomas elements

To illustrate vector-valued functions, we present here the next-to-lowest order Raviart-Thomas element on a triangle. This illustrates the essential features of our approach for vector elements, and can be extended to higher order approximating spaces with appropriate generalizations of the annihilators. We note that the Brezzi-Douglas-Marini elements [4],

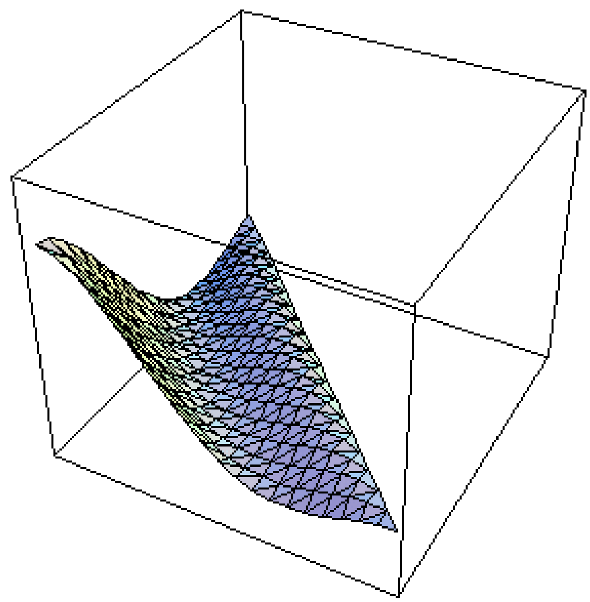

Figure 6: A nodal basis function for Hermite quartics

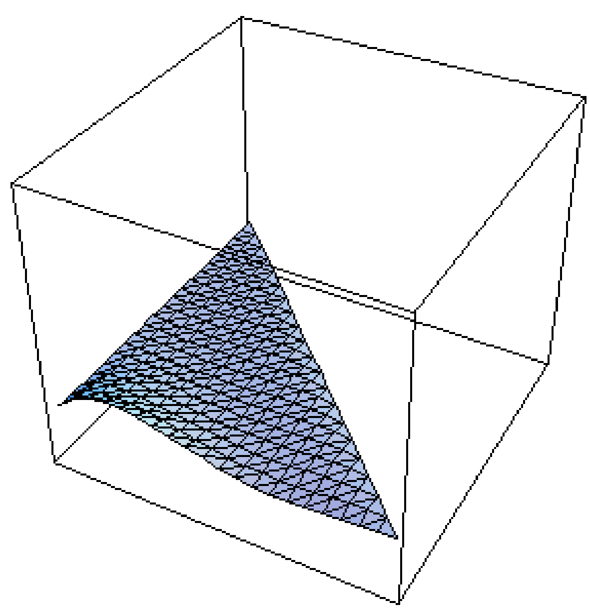

Figure 7: A nodal basis function for Hermite quartics

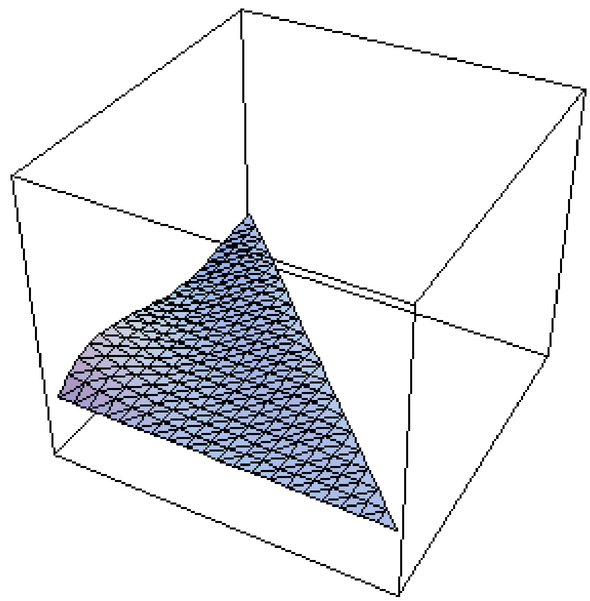

Figure 8: A nodal basis function for Hermite quartics

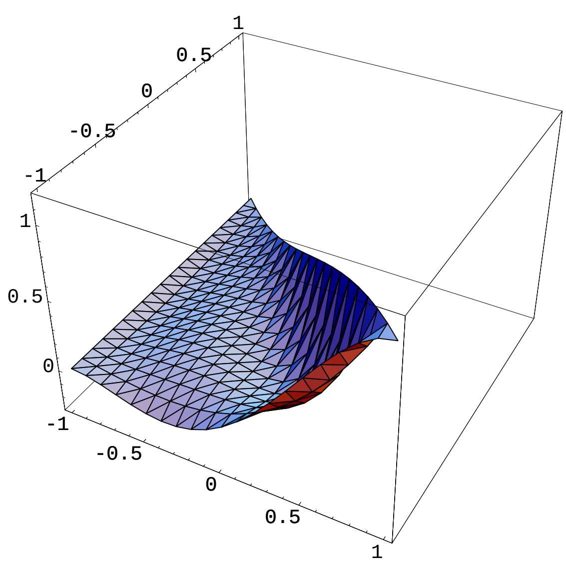

Figure 9: A nodal basis function for constrained Hermite quintics

while having more degrees of freedom than Raviart-Thomas, are conceptually much simpler since they are an unconstrained space. They have been handled in a separate work [10].

We follow the definition of Brezzi and Fortin [2] rather than the original Raviart-Thomas paper by defining the Raviart-Thomas space over a triangle K as

$$RT_k\left(K\right) = \left(P_k\left(K\right)\right)^2 + \tilde{x}P_k\left(K\right) \tag{59}$$

The dimension of RT_k is $(k+1)(k+3)$, and $\left(P_k\right)^2 \subset RT_k \subset \left(P_{k+1}\right)^2$.

We let $\{x_i\}_{i=1}^{3(k+1)}$ be points along the boundary of K with $k+1$ points on each edge. Then the nodes of RT_k are

- $\psi \cdot n\left(x_i\right), 1 \leq i \leq 3(k+1)$
- $\left(\psi, p\right)_K, p \in \left(P_{k-1}\right)^2$

In the case of RT_1, the last conditions merely consist of integration of each component against a constant. In higher order cases, we may, of course, use the scalar orthonormal basis, greatly simplifying the formation of the constraint and van der Monde matrices.

We characterize the special case of RT_1 in terms of annihilators as follows. More general results may be had. We note that RT_1 is the null space of the operators (acting on $\left(P_2\right)^2$)

- $\left(D_y^2, 0\right)$
- $\left(0, D_x^2\right)$
- $D_x\left(D_x, -D_y\right)$
- $D_y\left(D_x, -D_y\right)$

The first two operators insist that members of RT_1 have x components that are only linear in the y variable and y components that are linear in the x variable. This ensures that $\nabla \cdot RT_1 \subset P_1$. The second two operators require that the highest order terms consist of a scalar polynomial times the position vector.

We may cast these operators as block matrices using the expressions of differentiation found above. Then, we may compute the intersection of their null spaces by an SVD-based algorithm such as in Chapter 12 of [8]. This gives rise to the matrix $\mathcal{V}$, whence we obtain the orthonormal basis for RT_1. Finally, we proceed to compute the van der Monde matrix and tabulate the nodal basis at a collection of points.

Two of the nodal basis functions are shown in Figures 4.3 and 4.3. We choose our edge points to be $(-\frac{1}{3}, -1), (\frac{1}{3}, -1), (\frac{1}{3}, -\frac{1}{3}), (-\frac{1}{3}, \frac{1}{3}), (-1, \frac{1}{3}), (-1, -\frac{1}{3})$. The function in Figure

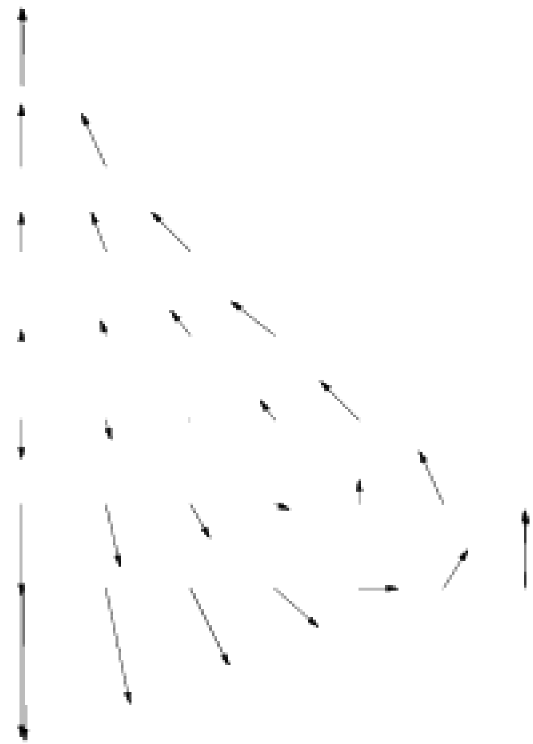

Figure 10: A nodal basis function for RT_1

4.3 has unit normal component at $(-\frac{1}{3}, -1)$, vanishing normal component at the other edge points, and each component vanishes against the constant Dubiner function. The function in Figure 4.3 has vanishing normal component at each of the edge points and, and the integral of its y component against constants also vanishes. Its x component integrated against $1/\sqrt{2}$ is 1.

5 Conclusion

Numerical linear algebra provides a computational language for an abstract class of finite elements. Nodal bases may be computed, and local bilinear forms evaluated through basic matrix computation. While we have focused on the element level, it is natural to use global assembly procedures to employ this approach for finite element methods. Software engineering centered around this paradigm could lead to very abstract and elegant, yet efficient, codes.

Besides the application of this methodology for particular circumstances, several open questions surround our use of van der Monde matrices. While this approach has been used

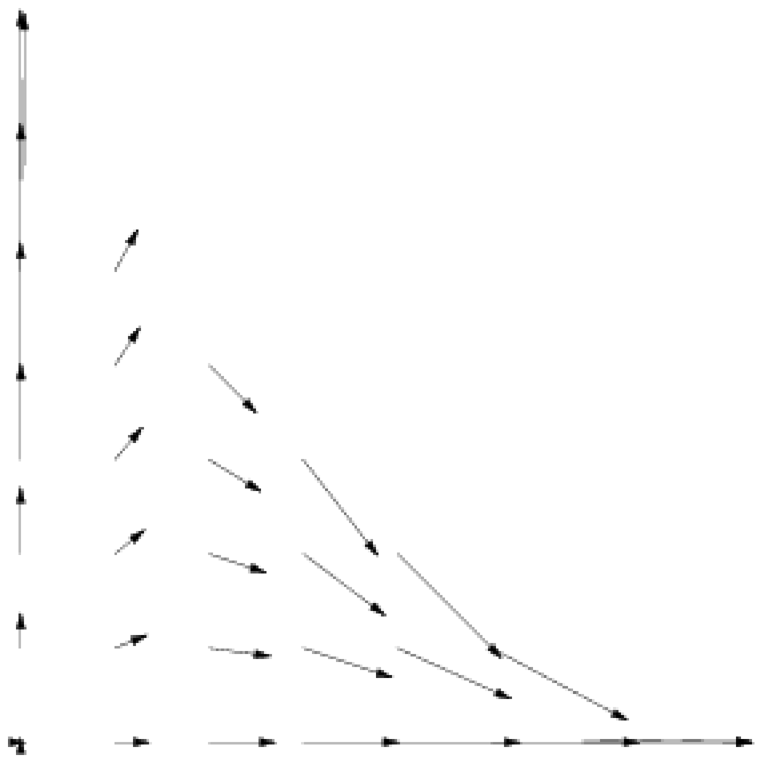

Figure 11: A nodal basis function for RT_1

for Brezzi-Douglas-Marini mixed elements in [10] and Lagrangian elements with the Fekete points in [9], a general result on the conditioning of the van der Monde matrices would be very useful. Further, efficient algorithms akin to those for standard van der Monde matrices would be a significant contribution.

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