RANKING IN AGGREGATE

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Abstract. We review several different approaches to rank aggregation for noisy and partial rankings. These techniques use different approaches to compute an optimal ranking. The algorithms considered are the Hungarian algorithm, HodgeRank (logarithmic least squares), and Analytic Hierarchy Process (principal eigenvector). We compare the rankings learned over a small wine dataset and a larger novel dataset of college rankings. Each ranking method is analyzed using quality metrics based on the underlying objective function which enables us to compare and contrast their performance on these datasets.

1. Introduction

Ranking is a broadly employed and beguilingly simple binary relation. The term “Rank” is the 3899th most used word in English [9], yet it can be an ambiguous term referring to either cardinal or ordinal relations. This is particularly true with aggregating multiple ranks, such as the two rankings \( (A = 1, B = 2, C = 3) \) and \( (C = 1, B = 2, A = 3) \) which can be considered either in complete disagreement or in agreement that \( B \) belongs in the “second” category. Linguistically we may illustrate the difference by saying either “\( A \) is better than \( B \) and \( B \) is better than \( C \),” or “\( A \) is ranked first, \( B \) is ranked second, and \( C \) is ranked third.” The two approaches to rank aggregation take opposing views, ordinal algorithms treat the relation as “\( A \) is better than \( B \)”, while categorical algorithms assign \( B \) to the “second” category. In this paper we will explore algorithms that treat ranking first as catagorical relations (Hungarian algorithm) and next as ordinal relations (HodgeRank, Analytic Hierarchy Protocol) in order to aggregate disagreements on unevenly labeled data.

Ranking is used in areas as diverse as social networks [8], voting [4], biological research [2], operations research [18], drug discovery [1], statistics [5, 6] and game theory [3]. Ranking algorithms have been consistently and extensively studied by a considerable number of professionals and academics. Computational advancements have breathed new life in the domain of ranking by enabling the number of items to be ranked and the number of raters to grow to immense size. Naturally, engineers would like to leverage these datasets, ideally with the results customized to individuals, and with as much current information as possible. To achieve these goals, they require computationally lightweight algorithms that do not require preprocessing.

These goals are often at odds with a great deal of historical statistical analysis. Indeed, statistics has a successful history of analyzing ranking data. Statisticians might sample a probability distribution from a representative population, then construct probability models with associated loss functions, and determine optimality after updates are performed. There is a rich history of statistical pursuit of this topic, and we are well aware of the criticisms levied against purely computational techniques especially when related to modeling [15]. There is much positive to be said regarding the traditional statistical results, yet often they do not scale easily. Theoreticians and engineers are attempting to reformulate traditional notions of statistics into a format that is capable of using multiple processors. [10]

Due to these constraints, among others, we limit ourselves to a different framework than described. We will often be thinking of situations when complex models of behavior are simply not known or are too costly to compute. Less computationally demanding algorithms are required in
order for the evaluation to be efficient. In such circumstances, numerically efficient computations offer powerful ways to compute solutions that might not otherwise be possible.

Our goal is to consider and make sense of what might historically have been considered “problematic” datasets. Such datasets might have the following properties:

1. **Unevenly Labeled Data**: Cases where raters label some data many times and other data rarely.
2. **Many Users/Items**: Case where the number of users and/or items is very large.
3. **Individual Ranking**: Case where users receive an individualized rank order.

Previous work has noted [11] that there are four aspects that typify our new internet scale datasets:

1. The data is often composed of cardinal scores.
2. The data is often incomplete.
3. Data collection is imbalanced.
4. The data can be represented as a network.

Several research communities have formed in an attempt to manage the unique characteristics of this data. They include:

- **Static Ranking**: Google must decide how to order a series of search results.
- **Rank Learning**: Netflix must learn the rank of movies for individual viewers.
- **Rank Aggregation**: When multiple people disagree on ranking order, one must find a way to appropriately aggregate their opinions.

The focus of this paper will be the algorithms that are designed to solve the problem of rank aggregation. This paper contributes to the previous work by providing a comparison of three algorithms on a new real world dataset. Broadly we separate our analysis into two different categories. In the first section we treat ranking as an aggregation of categorical labels. This natural interpretation allows the algorithms to utilize complete information when calculating an optimal assignment but comes with a computational cost. Naive implementation of such algorithms require management of $m!$ permutation of $m$ items and even when $m$ is of moderate size this is no longer computationally feasible. The second solution to this problem lies in the use of a $n \times n$ binary comparison matrix. Binary comparison algorithms don’t operate directly on the space of partial rankings or permutations rather they operate on the binary comparisons matrix. While computationally efficient this introduces a problem. All valid rankings can be turned into a unique and valid binary comparison matrix, but not all binary comparison matrices can be turned into a valid ranking. A consistent binary comparison matrix $Y$ is one that can be formed from one ranking vector $v$. Take for instance an all positive entry reciprocal matrix $Y$ formed as follows:

$$Y_{i,j} = \frac{v_i}{v_j}$$

or the alternative skew symmetric matrix

$$Y_{i,j} = v_j - v_i.$$  

Binary comparison matrices that represent the opinions of a number of different individuals are rarely consistent. The challenge presented is how should an algorithm choose the optimal consistent matrix from a given inconsistent binary comparison matrix. These algorithms leverage a graph theoretic interpretation of the ranking problem, where nodes represent items and edges weights reflect the preference for one item over another.

The solutions we explore approach the data from dramatically different perspectives; first an entirely disconnected graph, next the geometric mean of the graph as given, and finally an iterative approach where the algorithm converges to the stationary probability distribution as a result of
walk of infinite length. Finally we will compare the results and metrics for accuracy for each of the
three algorithms.

(1) **Hungarian Algorithm (HA):** The Hungarian algorithm treats the ranking data as cat-
egorical labels with the optimal ranking being the one that matches the largest number of
ratings for each category.

(2) **HodgeRank (HR):** HodgeRank utilizes Hodge decomposition to separate the binary com-
parison matrix into three separate orthogonal matrices (graduate, harmonic and curl flow). De-
spite the substantial theory involved in the result, the algorithm reduces computationally
to a logarithmic least squares projection.

(3) **Analytic Hierarchy Process (AHP):** AHP constructs a reciprocal binary comparison
matrix and by Perron–Frobenius theorem notes that the eigenvector corresponding to the
unique largest eigenvalue represents the dominance of each row item over each column.
AHP, and thereby the Perron–Frobenius theorem, was used extensively in the operation
research community since the early 1980’s.

2. Algorithms

The algorithms we consider take two fundamentally different ways of approaching ranking. Cat-
egorical rank aggregation determines and optimal ordering by finding the permutation that maxi-
mizes the correct number of labels given by raters. This amounts to the Linear Assignment problem
and we employ the Hungarian algorithm as a baseline for comparison. It does not utilize the orien-
tation of items until after the aggregation. Ordinal rank aggregation treats ranking data as partial
ordering of alternatives and employs a binary comparison matrix to deduce the optimal ordering.
Ordinal algorithms don’t operate direction the space of partial rankings, but rather operate on
binary comparison matrices. This is often interpreted as a graph where items as nodes, and \( a_{ij} \)
entry in the binary comparison matrix as the directed edges from item \( i \) to item \( j \). HodgeRanks
finds the geometric mean of the graph as presented, while AHP uses the largest eigenvector method
to smooth inconsistencies across an infinite walk on the graph.

2.1. Categorical: There are times when a ranking is best interpreted as a bijection from items to
labels, rather than a mapping of items to positions on the real line. Items are assigned to labels and
no relationship can be computed between labels. Each item should be assigned weights solely by
the most selected label. Counting the number of times an item received a particular position value
produces an item by position matrix \( Y \) that is the aggregation of multiple raters. The objective is
to find a permutation matrix \( P_{\sigma} \) that assigns the items to position in order to maximize the trace
\[
\hat{\sigma} = \arg \max_{\sigma \in S_n} \text{tr}(P_{\sigma}Y).
\]
where \( S_n \) is the set of permutations of symmetric group of size \( n \). A related problem, the Quadratic
Assignment problem, is NP-Hard yet some techniques for estimating the solution to this algorithm
rest upon the similar insight that the permutation matrix is drawn from \( S_n \) [12].

2.2. The Hungarian Algorithm. The original problem is often referred to as the “The As-
signed Problem” with a history going back as far as the 1890’s with Jacobi. We consider the
Hungarian algorithm [13] which was popularized in a 1955 paper by Harold Kuhn and predates
the Ford–Fulkerson algorithm [7] often used in computer science classes. While a naive algorithm,
it does allow us to preserve and exploit particular types of structure within the data. Consider
the following \( n \times m \) data matrix \( S \), where we consider a population of \( n \) raters \( r \) who with even
probability provide one of two ratings

\begin{equation}
S = \begin{bmatrix}
  t_1 & t_2 & t_3 \\
  r_1 & 1 & 2 & 3 \\
  r_2 & 3 & 2 & 1 
\end{bmatrix}.
\end{equation}

While there is disagreement on the position of \( t_1 \) and \( t_3 \), there is no disagreement on the ranking of \( t_2 \). It seems intuitive that \( t_2 \) always places second is important and worth preserving in the ranking data. We generate from each rater \( k \) a \( m \times m \) matrix where the rows represent items and the columns represent labels. Each entry is defined as:

\[ Y_{k,(i,j)} = \begin{cases} 
1 & \text{if } s_{ki} = j, \\
0 & \text{otherwise}. 
\end{cases} \]

We construct an aggregation matrix \( \bar{Y} \) for our minimization problem where each entry calculates the percentage disagreement among the voters for a particular item/label pair. It is calculated as follows:

\[ \bar{Y}_{i,j} = \frac{\sum_{k=1}^{n} Y_{k,(i,j)}}{\sum_{k=1}^{n} \sum_{j=1}^{m} Y_{k,(i,j)}}. \]

This matrix represents the percentage disagreement that each item received for each position normalized across the row. Using the Equation 1 would produce a matrix as follows

\begin{equation}
\bar{Y} = \begin{bmatrix}
  1 & 2 & 3 \\
  \cdot & 0 & \cdot \\
  \cdot & \cdot & \cdot 
\end{bmatrix}.
\end{equation}

The goal of the Hungarian algorithm is to find a maximal value by choosing one entry per row and column, thereby maximizing the percentage of votes accounted for. In the case of a tie, the algorithm simply randomly selects one option. Despite the simplicity of the algorithm we will see examples for which this is a natural and useful interpretation.

This example represents a ranking in which high prior is placed on having one of the two supplementary goods, but low priority on the complementary good. The complementary good is always preferred over the second supplementary good, but never preferred over both.

Consider the iconic American 4th of July picnic, where the options for food are hamburgers, hot dogs and chips. The celebrations would be incomplete without either hot dogs or hamburgers, and let us imagine preference between the two is evenly distributed. All would prefer chips over a second sandwich to complement their first choice. Our bijection of items on the left, and ranking on the right, would be constructed as follows:

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (h) at (0,0) {Hamburger};
  \node (c) at (0,-1.5) {Chips};
  \node (d) at (0,-3) {Hot Dog};
  \node (1) at (1,0) {$\cdot$};
  \node (2) at (1,-1.5) {$\cdot$};
  \node (3) at (1,-3) {$\cdot$};
  \draw[->] (1) edge node[above] {.5} (c);
  \draw[->] (2) edge node[above] {.5} (c);
  \draw[->] (3) edge node[above] {.5} (c);
  \draw[->] (1) edge node[above] {.5} (h);
  \draw[->] (2) edge node[above] {.5} (h);
  \draw[->] (3) edge node[above] {.5} (h);
\end{tikzpicture}
\caption{Set of edge weights from items to labels.}
\end{figure}

In our illustrative example, there were two separate communities that are only distinguishable by a second order statistic. In real data, this might take the form of a bimodal distribution for both hotdogs and hamburgers and a normal distribution for chips.
2.3. **Hungarian algorithm explanation.** Here we present the evaluation of the Hungarian algorithm for clarity as a minimization. For brevity we present an example that converges to an optimal in the first iteration. Begin with a matrix such as:

\[
Y = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
0.25 & 0.25 & 0 & 0.25 & 0.25 \\
0.1 & 0 & 0.1 & 0.7 & 0.1 \\
0.6 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.3 & 0.3 & 0.1 & 0.2 & 0.1 \\
0.3 & 0.4 & 0.1 & 0.1 & 0.1 
\end{bmatrix}
\]

Find the minimal value in each row and subtract it from that row:

\[
Y' = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
0.15 & 0.25 & 0 & 0.25 & 0.25 \\
0.1 & 0 & 0.1 & 0.7 & 0.1 \\
0.6 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.3 & 0.3 & 0.1 & 0.2 & 0.1 \\
0.3 & 0.4 & 0.1 & 0.1 & 0.1 
\end{bmatrix}
\]

Next find the minimal value in each column and subtract it from that column:

\[
Y'' = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
0.15 & 0.25 & 0 & 0.25 & 0.25 \\
0 & 0 & 0 & 0.7 & 0.1 \\
0.4 & 0 & 0 & 0 & 0 \\
0.1 & 0.2 & 0 & 0.1 & 0 \\
0.1 & 0.3 & 0 & 0 & 0 
\end{bmatrix}
\]

In this step, look for a bijection from rows to columns considering only entry equal to zero in the \(Y''\) matrix. If one exists, as one does in this example, return it. In our example \(Y''\) we have underlined the appropriate bijection, which results in a value of 0.4. For brevity we will not consider the case where it does not see ref [13] for the full explanation. Despite the simplicity of this algorithm, it is
able to identify a type of structure within the data that leads it to be surprisingly consistent with the more sophisticated algorithms we consider. One reason for this is implied in a later section by Figure 6. There it is apparent that if each item’s proposed labels are drawn from a single Gaussian distribution centered on a unique label then Hungarian Algorithm will perform well. For a non-trivial set of data we consider this turns out to be the case.

3. Binary Comparison Matrices

The use of paired or binary comparisons in statistical ranking dates at least as far back as Zermelo in 1929 [22]. Here we present the options available when constructing binary comparison matrices as these formulations will appear repeatedly throughout the rest of the work. Whenever possible we use the standard notation convention. Matrices are denoted by uppercase Roman letters while vectors are lowercase Roman. Scalars are denoted with unbolded Roman letters. Let $\mathbb{1}$ denote a vector of all ones. Unless otherwise specified we will abide by the conventions in our discussion.

For our purposes we will begin with a set $R$ of raters who provide judgments on a set of $T$ items. This is our data matrix and typically looks as follows:

$$ S = \begin{bmatrix}
  t_1 & t_2 & t_3 & \cdots & t_m \\
 r_1 & s_{1,1} & s_{1,2} & \cdots & s_{1,m} \\
 r_2 & s_{2,1} & s_{2,2} & \cdots & s_{2,m} \\
 r_3 & s_{3,1} & s_{3,2} & \cdots & s_{3,m} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 r_n & s_{n,1} & s_{n,2} & \cdots & s_{n,m}
\end{bmatrix} $$

where we define the set of raters $R = \{r_1, \ldots, r_n\}$ and the set of items are defined as $T = \{t_1, \ldots, t_m\}$ where each $r_i$ and $t_i$ represents an individual rater and item respectively. $S$ may not be a dense matrix. Often individuals only rate a very small subset of items. Define the $r_i$ as the vector of $r_i$’s rating for the $t_j$ item as $r_i = \{s_{i,1}, \ldots, s_{i,m}\}$.

To construct a binary comparison matrix for each rater, we must specify a function to compare every two items for each entry in the matrix.

$$ f(s_{i,j}, s_{i,k}) $$

as a function that takes two ratings from one rater,

$$ Y_z $$

to be an item by item pairwise comparison matrix of size $m \times m$ for rater $r_z$,

$$ Y_{z,(i,j)} $$

according to a function $f(s_{z,i}, s_{z,j})$.

This allows us to construct for each rater $r_z$ a binary comparison matrix $Y_z$ as follows:

$$ Y_z = \begin{bmatrix}
  t_1 & t_2 & t_3 & \cdots & t_m \\
 t_1 & f(s_{z,1}, s_{z,1}) & f(s_{z,2}, s_{z,1}) & \cdots & f(s_{z,m}, s_{z,1}) \\
 t_2 & f(s_{z,1}, s_{z,2}) & f(s_{z,2}, s_{z,2}) & \cdots & f(s_{z,m}, s_{z,2}) \\
 t_3 & f(s_{z,1}, s_{z,3}) & f(s_{z,2}, s_{z,3}) & \cdots & f(s_{z,m}, s_{z,3}) \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 t_m & f(s_{z,1}, s_{z,m}) & f(s_{z,2}, s_{z,m}) & \cdots & f(s_{z,m}, s_{z,m})
\end{bmatrix} $$

This matrix represents a single rater’s paired comparison of all the items they evaluated. Since our goal is the aggregation of multiple raters into one global ranking we then move to the second portion.
of the algorithm. Here we aggregate multiple $Y_i$ matrices into a single matrix $\bar{Y}$ that reflects the agreement and disagreement of the raters. Next define:

$$Y = \{Y_1, \ldots, Y_n\} \quad \text{to be the set of all single rater pairwise comparison graphs,}$$

$$\bar{Y} = F(Y) \quad \text{to be a function that aggregates } \{Y_1, \ldots, Y_n\} \text{ to form } \bar{Y}.$$ 

Our final binary comparison matrix $\bar{Y}$ provides aggregated comparison between all items, and across all raters. We pause to note that the choice of $f$ and $F$ have considerable impact on our results in $\bar{Y}$ and should be chosen to reflect the problem being considered. The evaluation functions described below (ratio of difference, magnitude of difference and binary comparison) reflect varying interpretations of the original data. While extensive analysis on real world data is beyond the scope of this paper we a sample of typical of comparison and aggregation functions.

3.0.1. *Ratio of difference.* Applicable to both scores and rankings, a considerable portion of our analysis will consider matrices of this form

$$(4) \quad f(s_{k,i}, s_{k,j}) = \frac{s_{k,j}}{s_{k,i}}.$$ 

3.0.2. *Magnitude of difference.* This may be applied to either a ranking, or a score. In the case of the ranking, the resulting value indicates the difference in position between item $i$ and item $j$. In the case of a score, it returns the score difference.

$$(5) \quad f(s_{k,i}, s_{k,j}) = s_{k,j} - s_{k,i}.$$ 

3.0.3. *Binary comparison.* Defined as

$$(6) \quad Y_{i(j,k)} = \text{sign}(s_{i,k} - s_{i,j})$$

or identically

$$r_{(k)ij} = \begin{cases} 1 & \text{if } j > i, \\ -1 & \text{if } j < i, \\ 0 & \text{if } b = a. \end{cases}$$

3.0.4. *Matrix form.* The evaluation functions above can result in either a reciprocal or a skew symmetric matrix. A skew symmetric matrix takes the form

$$Y_{ij} = -Y_{ji}.$$ 

A reciprocal matrix, takes the form:

$$Y_{ij} = \frac{1}{Y_{ji}}.$$ 

We note that switching from one structure to the other is as straight forward as exponentiating or logging the matrices. If we begin with a reciprocal matrix such as

$$s_{k,(i,j)} = \frac{s_{k,j}}{s_{k,i}}.$$ 

We may convert it to a skew symmetric by

$$\log s_{k,(i,j)} = \log s_{k,j} - \log s_{k,i}$$

and convert it back by exponentiating. Regardless of the form of our matrix we need to aggregate the individual rater matrices into a single aggregate binary comparison matrix $\bar{Y} = F(Y)$. Our
approach to aggregating individual rater's matrices differs only according to the structure of the matrices. In the case of skew symmetric matrices

\[ Y_{(j,k)} = \sum_{i=1}^{n} (Y_{i(jk)}). \]

When considering reciprocal matrices we multiply every entry in the same position \( i, j \) across as matrices.

\[ Y_{(j,k)} = \prod_{i=1}^{n} (Y_{i(jk)}). \]

To reiterate a central point, irrespective of the structure the principle insight that leads to the use of binary comparison matrices is that if \( Y \) has a single vector \( v \) ranking all the items then the matrix is defined as consistent. If when considering skew symmetric matrices, consistent matrices are of the form

\[ Y = v\mathbb{1}^\top - \mathbb{1}v^\top \]

where \( \mathbb{1} = [1, \ldots, 1]^\top \). The ultimate goal of our ranking algorithms is to find a “optimal” consistent matrix denoted as \( Y_{\text{opt}} \) when presented with a properly constructed \( Y \).

Returning to Equation 1, it is natural to note that \( r_1 \) and \( r_2 \) disagree fundamentally about the ordering. In everyday language we might say that they were opposites, reversed or that they disagree completely. The reason for this is that we are considering not just the position of each item, but rather the relationship between the items. We can see this if we consider the binary comparison matrices constructed using equation 1. The individual matrices are the transpose of one another:

\[
Y_1 = \begin{bmatrix}
t_1 & t_2 & t_3 \\
t_2 & 0 & 1 & 2 \\
t_3 & -1 & 0 & 1 \\
\end{bmatrix}
\]

\[
Y_2 = \begin{bmatrix}
t_1 & t_2 & t_3 \\
t_2 & 0 & -1 & -2 \\
t_3 & 1 & 0 & -1 \\
\end{bmatrix}
\]

when summed would yield

\[
Y = \begin{bmatrix}
t_1 & t_2 & t_3 \\
t_2 & 0 & 0 & 0 \\
t_3 & 0 & 0 & 0 \\
\end{bmatrix}
\]

This coincides with our intuition, when thinking of ordering, that \( r_1 \) and \( r_2 \) offered completely opposite rankings and therefore nothing can be deduced about their relationships. The common vernacular reflects the inherent relationship we often imply when performing ranking. Namely that ranks are more than labels, they are an ordering that allows comparison between items.

One common approach that these algorithms take is to consider the binary comparison matrix \( Y \) as a graph \( G \) composed of ordered pairs \((V, E)\). Where \( V \), the vertex set, is a finite set and \( E \), edge set, is an unordered two element \((v_i, v_j)\) pair where \( v_i \) and \( v_j \) are elements of \( V \). Vertices \( v_i \) and \( v_j \) are considered adjacent if there exists some edge \( E = (v_i, v_j) \). The goal of a the binary comparison algorithm is to remove intransitives from the aggregate matrix \( Y \). HodgeRank and AHP uses different approaches, which we will evaluate more closely in a later section, however for the moment consider this principle difference. HodgeRank computes the optimal on only the graph edge set as given. If the binary comparison matrix is a reciprocal matrix, then HodgeRank computationally amounts to a Logarithmic Least Squares or identically the Geometric Mean of the provided graph. AHP compares the dominance of each item over every other item thru infinite path lengths. A difference in ranking between these two algorithms is referred to as a rank reversal [19].
Figure 3 shows such an example, where one item dominates the explicit graph edges (i.e. paths) as presented, while another item dominates when there is an iterative calculation across all possible paths lengths. Figure 3 presents an example where every path length equals one, while 4 presents an example where edges have weights.

This graph represents the ordering to two raters, where \( r_1 \) produces the following ordering \( t_1 > t_2 > t_3 > t_4 > t_5 > t_6 > t_7 > t_8 \) and \( r_2 \) rated items \( t_9 > t_{10} > t_{11} > t_{12} > t_1 \). In this example we will concern ourselves with item’s \( t_1 \) and \( t_9 \) as they represent two very different ways of considering the optimal ranking. Accordingly, when considering only those paths explicitly labeled on our graph \( t_9 \) ranks above \( t_1 \). When considering AHP and an infinite path length, item \( t_1 \) ranks above item \( t_7 \) due to the larger cluster size that \( t_1 \) dominates.

Table 1. HodgeRank and AHP computed on Figure 3.

<table>
<thead>
<tr>
<th></th>
<th>HR Score</th>
<th>Rank</th>
<th>AHP Score</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(-.25)</td>
<td>5</td>
<td>(-.387)</td>
<td>1</td>
</tr>
<tr>
<td>(t_1)</td>
<td>1</td>
<td>6</td>
<td>(-.378)</td>
<td>3</td>
</tr>
<tr>
<td>(t_2)</td>
<td>(-.5)</td>
<td>7</td>
<td>(-.324)</td>
<td>5</td>
</tr>
<tr>
<td>(t_3)</td>
<td>.75</td>
<td>8</td>
<td>(-.275)</td>
<td>7</td>
</tr>
<tr>
<td>(t_4)</td>
<td>1</td>
<td>9</td>
<td>(-.235)</td>
<td>9</td>
</tr>
<tr>
<td>(t_5)</td>
<td>1.25</td>
<td>10</td>
<td>(-.200)</td>
<td>11</td>
</tr>
<tr>
<td>(t_6)</td>
<td>1.5</td>
<td>12</td>
<td>(-.171)</td>
<td>12</td>
</tr>
<tr>
<td>(t_7)</td>
<td>(-1.85)</td>
<td>2</td>
<td>(-.146)</td>
<td>2</td>
</tr>
<tr>
<td>(t_8)</td>
<td>(-1.45)</td>
<td>3</td>
<td>(-.380)</td>
<td>4</td>
</tr>
<tr>
<td>(t_9)</td>
<td>(-1.45)</td>
<td>4</td>
<td>(-.324)</td>
<td>6</td>
</tr>
<tr>
<td>(t_{10})</td>
<td>(-1.5)</td>
<td>8</td>
<td>(-.277)</td>
<td>8</td>
</tr>
<tr>
<td>(t_{11})</td>
<td>(-.65)</td>
<td></td>
<td>(-.23)</td>
<td></td>
</tr>
</tbody>
</table>

Another example from [20] considers weighted edges and notes how the structure and weight of the edge flow leads to rank reversal in the following matrix:

\[
S = \begin{bmatrix}
    t_1 & t_2 & t_3 & t_4 & t_5 \\
0 & 1/7 & 1/2 & 1/8 & 2 \\
1 & 7 & 1 & 3 & 1 & 8 \\
2 & 1/3 & 1 & 4 & 5 & 1/4 & 3 & 1/8 & 1/5 & 1/5 & 1
\end{bmatrix}
\] (9)

This is typically depicted in Figure 4 where it is customary to omit the edge weights of 1. In this case, the ranking order alternates from \( t_2 > t_4 > t_3 > t_1 > t_5 \) in the case of HodgeRank to \( t_4 > t_2 > t_3 > t_1 > t_5 \) when using AHP.
Table 2. HodgeRank and AHP computed on Matrix 9 and Figure 4 from [20].

<table>
<thead>
<tr>
<th></th>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( t_3 )</th>
<th>( t_4 )</th>
<th>( t_5 )</th>
</tr>
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<td>Score</td>
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<td>.384</td>
<td>.133</td>
<td>.380</td>
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<td></td>
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<td>1</td>
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<td>2</td>
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<tr>
<td>AHP</td>
<td>Score</td>
<td>61</td>
<td>.374</td>
<td>.134</td>
<td>.387</td>
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<tr>
<td></td>
<td>Rank</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Figure 4. Directed Graph representing matrix 9

3.1. **HodgeRank**. HodgeRank, as noted above, uses Hodge theory to remove intransitives from an inconsistent binary comparison matrix. When presented with such data, HodgeRank finds the closest consistent matrix using the \( L_2 \) norm. Computationally this amounts to the logarithmic least squares approximation of a reciprocal matrix. Hodge theory in general provides a decomposition into the following orthogonal subspaces: *gradient flow*, *harmonic flow* and *curl flow*. These spaces have clear interpretations in our graph theoretic construction. *Gradient flow* that is acyclic globally, that is a consistent binary comparison matrix. While a *harmonic flow* is globally cyclic where there is not clear beginning or end such as the ordering \( a > b > c > a \). Finally, *curl flow* is locally cyclic, such as the ordering \( a > b > c > d > b > e \). Items \( b,c,d \), are cyclic, while items \( a \) and \( e \) belong in a particular position. Stated differently the acyclic gradient flow, denoted with \( C \), is optimal ranking, while the sum of the the harmonic and curl flow, defined as a matrix \( U \), represent the intransitives within the matrix. The magnitude of \( U \) allows the evaluation of the accuracy of optimal \( C \) approximation of the original data. Separating the curl from the harmonic flow allows for further analysis. If curl flow dominates the harmonic flow, then local inconsistencies are more dominant implying there is trouble ordering closely ranked items. It some situations this could be interpreted as small clusters of equivalent objects. This in itself is a novel contribution from HodgeRank as no other algorithms provide a clear interpretation of the inconsistencies identified.

*Gradient*, *harmonic* and *curl* flow are all skew symmetric matrices. The gradient flow is defined as when \( i,j \in E \), the gradient is:

\[
(\text{grad } s)(i,j) = s_j - s_i.
\]

While the curl is the edge flow \((Z)\) of a triangle formed from three vertices on graph \( G = (V,E) \), where \( e_{ij}, e_{jk}, e_{ki} \in E \) defined as:

\[
(\text{curl } Z)(i,j,k) = \begin{cases} 
Z_{ij} + Z_{jk} + Z_{ki} & \text{if } i,j,k \in E, \\
0 & \text{otherwise}.
\end{cases}
\]
Finally, harmonic flow represents the global inconsistencies and it measures the total flow across every item.

\[(\text{harmonic } G)(i) = \begin{cases} \sum Z_{ij} & \text{for all } i, j \in E, \\ 0 & \text{otherwise.} \end{cases} \]

3.1.1. Explanation of HodgeRank in matrix formation. While HR provides a robust mathematical theory, the focus here is largely upon the matrix interpretation. Any matrix \( A \) may be constructed uniquely from the sum of a skew symmetric and a symmetric matrix. Let us understand \( Y \) as composed of two separate matrices. \( Y = C + U \) where \( C \) signifies a consistent skew symmetric matrix, while \( U \) contains the inconstancies. Since \( C \) is skew symmetric we can represent it as

\[ C = r \mathbf{1}^\top - \mathbf{1} r^\top \]

where \( r \) indicates a vector of the items. We would like to choose the smallest \( U \), according to its Frobenius norm, such that \( C \) is consistent and skew symmetric. To achieve this we do the following

\[ \min \|U\|_F^2 = \min \| Y - C \|_F^2. \]

Which by substitution can be represented as

\[ \min \|U\|_F^2 = \min_{b \in \mathbb{R}^n} \| Y - r \mathbf{1}^\top + \mathbf{1} r^\top \|_F^2. \]
Now we note that we have reduced our minimization to a choice of the optimal $r$. Consider the relationship between

$$\frac{\|C_l\|_F^2}{\|Y\|_F^2} \text{ and } \frac{\|U_m\|_F^2}{\|Y\|_F^2},$$

if

$$\frac{\|C_l\|_F^2}{\|Y\|_F^2} \gg \frac{\|U_m\|_F^2}{\|Y\|_F^2},$$

then we may deduce that the ratings are largely consistent and there are few cycles in the data. Otherwise,

$$\frac{\|C_l\|_F^2}{\|Y\|_F^2} \ll \frac{\|U_m\|_F^2}{\|Y\|_F^2}$$

which means there are many more inconsistencies in the data and we may adjust our perception of the ranking accordingly.

3.2. **Analytic Hierarchy Process (AHP).** Individuals and groups confronting tough decisions often find themselves in a position of conflicted opinions. Arrow’s famous impossibility theorem has shown that when sufficiently complex social preferences are aggregated transitivities are always possible. Analytic Hierarchy Process is a technique that was designed for one or more individuals to evaluate a complicated ranking problem with multiple criteria that was likely to generate inconsistencies when aggregated into a global ranking. The techniques we consider fall broadly within the literature related to the largest eigenvalue and its corresponding eigenvector.

The premise for such aggregation is a simple one. Suppose two groups of items are ranked as follows (zeros represent no ranking provided)

$$S = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ r_1 & 1 & 2 & 3 & 0 & 0 \\ r_2 & 0 & 0 & 1 & 2 & 0 \\ r_3 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}. $$

We note that it is natural to think of a transitivity from $r_1$’s ranking through $r_2$’s to $r_3$’s. We form this into binary comparison matrix as in Equation (3.0.3)

$$Y = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ t_1 & 0 & 1 & 1 & 0 & 0 \\ t_2 & -1 & 0 & 1 & 0 & 0 \\ t_3 & -1 & -1 & 0 & 1 & 0 \\ t_4 & 0 & 0 & -1 & 0 & 1 \\ t_5 & 0 & 0 & 0 & -1 & 1 \\ t_6 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}. $$

It seems logical to conclude that the global ordering should be $t_1 > t_2 > t_3 > \cdots > t_6$. However it is often the case that there are disagreements such as the following matrix

$$S = \begin{bmatrix} t_1 & t_2 & t_3 \\ r_1 & 1 & 2 \\ r_2 & 0 & 1 \\ r_3 & 2 & 0 \end{bmatrix}. $$

Here a global intransitivity appears ($t_1 > t_2 > t_3 > t_1$) in our binary comparison matrix. This is the worst possible case, as it is a perfectly even distribution across the 3 elements. The goal of all the algorithms in this section attempt to remove intransitive sections from the original $Y$ and thereby derive an optimal global ranking for all uses and items. Additionally these global
approximations often provide a way of characterizing the difference between the original dataset and the $Y_{opt}$ approximation.

Analytic Hierarchy Process, an algorithm developed in the operations research community, draws from mathematical insights, social choice theory, and sociological research. Described as a “theory of measurement” by its inventor Thomas Saaty in the late 1970s, AHP not only provides mathematical models for aggregation but additionally provides guidelines for the questioning process. The central mathematical result utilized is the Perron–Frobenius theorem, however the algorithm is more complicated. Saaty spent considerable time and effort to ensure that AHP would provide justifiable results for the operations research community.

Roughly AHP can be categorized as the following five steps:

1. Guide an individual or group in the judgement of items
2. Aggregate their responses
3. Calculate the consistency
4. Produce an optimal ranking
5. Refine if required

3.2.1. Data Collection. The AHP framework provides strict restrictions on the data collection process. Saaty, in a paper titled “Why the Magic Number Seven Plus or Minus Two”, [16] argues that the raters should be asked to review no more than $7(\pm 2)$ items at one time. Larger collections of items should be subdivided to observe this rule. This recommendation is in response to “Miller’s Law”, a 1956 result by G.A. Miller [14].

Saaty recommends using a multiplicative scale for user responses. Questions and answerers should denote the degree of dominance of item A over item B. The scale has the numerical values $1/9, 1/8, 1/7, \ldots, 1, 2, 3, \ldots, 9$ which are mapped to linguistic judgments such as “moderately more dominant”. This rating would imply that the “moderately more dominant” item is 3 times better than the less dominant one. Other descriptions include “strongly more dominant”, “very strongly more dominant”, and “extremely more dominant” which are given the ratings of 5, 7, 9 respectively. The intermediate values provide additional granularity for the rater. This approach allows him to populate the binary comparison matrix directly.

Saaty argues strongly for such structure, believing it prevents unnecessary intransitives from being introduced in the data collection process. This is particularly important when considering his consistency ratio, a calculation that is used to evaluate the accuracy of the original comparison matrix. While Saaty feels strongly regarding these points, we would be remiss not to point out that there is a wealth of research since 1956, both empirical and theoretical, that indicates that scales other than those presented here maybe superior. In our analysis we are forced to adapt his approach as we are unable to use his proscribed regimen but provide it for completeness.

3.2.2. Aggregation of data. For decisions that require multiple criteria, Saaty recommends the subdivision into manageable categories and then the reassembly of the material accordingly. [17] As a case example here we present a classification of whiskey based on its flavor profile. The dataset classifies 86 whiskeys, according to twelve criteria (Body, Sweetness, Smoky, Medicinal, Tobacco, Honey, Spicy, Winey, Nutty, Malty, Fruity, and Floral). Each whiskey was rated on a scale from 1-5, with a higher number indicating a stronger amount.

Suppose you wished to try the most whiskeys with the strongest flavors, and wished to use AHP to aggregate the above judgements. AHP constructs a pairwise comparison matrix from each trait. For instance the Body matrix looks as follows.

Next AHP calculates the principle eigenvector for each of the flavor profiles.
Table 3. Whiskeys and their ratings by flavor profile

<table>
<thead>
<tr>
<th>Criteria</th>
<th>B</th>
<th>Sw</th>
<th>Sm</th>
<th>M</th>
<th>T</th>
<th>H</th>
<th>Sp</th>
<th>W</th>
<th>N</th>
<th>Ma</th>
<th>Fr</th>
<th>Fl</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aberfeldy</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Aberlour</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>AnCnoc</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Ardbeg</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Ardmore</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Tullibardine</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Binary comparison table of whiskeys according to their “Body”.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>B</th>
<th>Sw</th>
<th>Sm</th>
<th>M</th>
<th>T</th>
<th>H</th>
<th>Sp</th>
<th>W</th>
<th>N</th>
<th>Ma</th>
<th>Fr</th>
<th>Fl</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aberfeldy</td>
<td>.105</td>
<td>.115</td>
<td>.91</td>
<td>.99</td>
<td>.11</td>
<td>.84</td>
<td>.117</td>
<td>.78</td>
<td>.89</td>
<td>.99</td>
<td>.98</td>
<td>.95</td>
</tr>
<tr>
<td>Aberlour</td>
<td>.77</td>
<td>.121</td>
<td>.66</td>
<td>.11</td>
<td>.28</td>
<td>.58</td>
<td>.78</td>
<td>.89</td>
<td>.66</td>
<td>.66</td>
<td>.95</td>
<td></td>
</tr>
<tr>
<td>AnCnoc</td>
<td>.14</td>
<td>.77</td>
<td>.91</td>
<td>.99</td>
<td>.11</td>
<td>.84</td>
<td>.146</td>
<td>.87</td>
<td>.131</td>
<td>.89</td>
<td>.99</td>
<td>.66</td>
</tr>
<tr>
<td>Ardbeg</td>
<td>35</td>
<td>.154</td>
<td>3</td>
<td>99</td>
<td>.11</td>
<td>.14</td>
<td>.87</td>
<td>.131</td>
<td>.119</td>
<td>.99</td>
<td>.131</td>
<td>.158</td>
</tr>
<tr>
<td>Ardmore</td>
<td>.105</td>
<td>.115</td>
<td>.91</td>
<td>.66</td>
<td>.11</td>
<td>.112</td>
<td>.117</td>
<td>.104</td>
<td>.89</td>
<td>.66</td>
<td>.131</td>
<td>.126</td>
</tr>
<tr>
<td>Tullibardine</td>
<td>.105</td>
<td>.76</td>
<td>.151</td>
<td>.99</td>
<td>.88</td>
<td>.14</td>
<td>.87</td>
<td>.104</td>
<td>.118</td>
<td>.99</td>
<td>.98</td>
<td>.126</td>
</tr>
</tbody>
</table>

Table 5. Normalized vectors of whiskey characteristics.

AHP notes that it is possible that not all traits are equally important. Rather than simply summing each of the traits, we may aggregate them according to their relative importance. If one wanted to buy a different whiskey with a similar profile to Lagaluvin, then one would use the feature vector for Lagaluvin and evaluate the matrix below. Such as:

<table>
<thead>
<tr>
<th>Criteria</th>
<th>B</th>
<th>Sw</th>
<th>Sm</th>
<th>M</th>
<th>T</th>
<th>H</th>
<th>Sp</th>
<th>W</th>
<th>N</th>
<th>Ma</th>
<th>Fr</th>
<th>Fl</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lagavulin</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 6. Profile of Lagaluvin

The algorithm simply computes that dominance of each feature for the desired whiskey and multiplies it against a normalized matrix from above. This weights the dominance of each whiskey according to how similar it is to the desired profile. The top result is Lagavulin of course but then Ardmore, Laphroaig, Talisker, Caol Ila, and Clynelish. In each case the feature vector for the
whiskey favor three values, Smokey, Medicinal, and Body. It is not necessary to choose an exact match or to have only a single individual specifying the options. Additionally when contending with very large datasets, AHP provides a mechanism to divide and evaluate portions of the data, and then aggregate in a hierarchical fashion. This is a very useful approach, because simply computing the “most flavorful” whiskey is not a very insightful evaluation.

3.2.3. Consistency Ratio. It is often the case that the binary comparison matrix is riddled with inconsistencies above leads to an inconsistent graph, i.e., one in which \( a_{ij} \times a_{jk} \neq a_{ik} \). In order to gauge the degree of consistency AHP calculates a consistency index for the data matrix defined as:

\[
CI = \frac{\lambda_{\text{max}} - n}{n - 1}.
\]

This is justified because it is known that when \( A \) is a consistent matrix with maximal eigenvalue \( \lambda_{\text{max}} = n \) where \( n \) is the dimension of the matrix. When \( A \) is inconsistent then \( \lambda_{\text{max}} > n \). As such \( CI \) provides a metric of the consistency of our provided matrix. The next step is to compute how different this consistency is from a randomly selected matrix of the same dimensions.

Next a table of the average \( \lambda_{\text{max}} \) values for different sized matrices is generated. This referred to as random index (RI). Saaty placed unique restrictions on the matrix size and matrix entries (as we have seen), which allowed him to generate the following table of random index for a matrix of a given size \( n \). [16]

Saaty’s objective is to compare the inconsistencies within the data matrix and the randomly generated matrix. To do so he normalizes the \( CI \) by the RI to generate a consistency ratio (CR)

\[
CR = \frac{CI}{RI}
\]

Saaty concluded that only matrices with a \( CR \leq 10\% \) may be used for analysis. This heuristic represents the trade off he perceived for consistency in operations research. He determined that if the \( CR > 10\% \) then the provided data matrix has too many intransitivities to produce a reliable result. In such cases, his recommendation is to return to the group or individual and ask them to clarify or otherwise alter their original answers. To facilitate this he generated a list of cut off values for an acceptable CR value presented here.

3.2.4. Optimal ranking. In our example all entries \( a_{ij} \) are strictly larger then 0, the Perron–Frobenius theorem tells us that there exists a largest unique positive eigenvalue such that

\[
|\lambda_{\text{max}}| > |\lambda_i| \text{ for all } i \neq \text{ max}.
\]

Additionally Perron–Frobenius theorem states that the eigenvector corresponding to the \( \lambda_{\text{max}} \) takes the form \( w = w_1, \ldots, w_n \), with all \( w_i \)'s greater than zero and real.

AHP has been wildly successful in the operations research community, spawning a tremendous number of publications and broadly implemented.

4. 1976 Wine Ranking dataset

For our initial analysis we consider a dense matrix of evaluation from 11 European wine experts in 1976. The eleven judges considered ten wines, six Californian and four French, in a blind taste

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( CR )</td>
<td>.52</td>
<td>.89</td>
<td>1.11</td>
<td>1.25</td>
<td>1.35</td>
<td>1.40</td>
<td>1.45</td>
<td>1.49</td>
<td>1.52</td>
<td>1.54</td>
<td>1.56</td>
<td>1.58</td>
<td>1.59</td>
</tr>
</tbody>
</table>

Table 7

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( CR )</td>
<td>.52</td>
<td>.89</td>
<td>1.11</td>
<td>1.25</td>
<td>1.35</td>
<td>1.40</td>
<td>1.45</td>
<td>1.49</td>
<td>1.52</td>
<td>1.54</td>
<td>1.56</td>
<td>1.58</td>
<td>1.59</td>
</tr>
</tbody>
</table>
testing. Their results were a shock to the wine industry as California wines outperformed several classic French Bordeaux. The original results were calculated by the host of the event, Steven Spurrier, a British reseller of French wine. The original rankings were established by an average score by summing all eleven ratings for each wine and dividing by 11.

Spurrier later recognized, as did many statisticians, that his calculated ranking offered little statistical guarantees and therefore the much lauded success of American wine over French was called into question. Spurrier had expected, and many wine experts agreed, that the American wines would place far below their French counterparts.

Spurrier, collected eleven individuals (Brejoux (B), Dovaz (D), Gallagher (G), Kahn (K), Millot (M), Oliver (O), Spurrier (S), Tari (T), Vanneque (Van), Villaine (Vil), Vrinat (Vri), and asked them to blindly sample 10 wines. Each was requested to rate the wines on a scale from 1–20, with 20 being the highest rating.

Here are their fully tabulated scores with the raters last names abbreviated:

<table>
<thead>
<tr>
<th>Wine</th>
<th>Total</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stag’s Leap</td>
<td>14.13</td>
<td>1</td>
</tr>
<tr>
<td>Mouton–Rothschild</td>
<td>14.9</td>
<td>2</td>
</tr>
<tr>
<td>Chateau Montrose</td>
<td>13.6</td>
<td>3</td>
</tr>
<tr>
<td>Haut–Brion</td>
<td>13.2</td>
<td>4</td>
</tr>
<tr>
<td>Ridge Vineyards</td>
<td>12.1</td>
<td>5</td>
</tr>
<tr>
<td>Leoville Las Cas</td>
<td>11.1</td>
<td>6</td>
</tr>
<tr>
<td>Heitz Wine</td>
<td>10.3</td>
<td>7</td>
</tr>
<tr>
<td>Clos Du Val</td>
<td>10.1</td>
<td>8</td>
</tr>
<tr>
<td>Mayacamas</td>
<td>9.9</td>
<td>9</td>
</tr>
<tr>
<td>Freemark Abbey</td>
<td>9.4</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rater</th>
<th>Average Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brejoux(B)</td>
<td>12</td>
</tr>
<tr>
<td>Dovaz(D)</td>
<td>11.5</td>
</tr>
<tr>
<td>Gallagher(G)</td>
<td>13.9</td>
</tr>
<tr>
<td>Kahn(K)</td>
<td>9.2</td>
</tr>
<tr>
<td>Millot(M)</td>
<td>11.6</td>
</tr>
<tr>
<td>Oliver(O)</td>
<td>11.6</td>
</tr>
<tr>
<td>Spurrier(S)</td>
<td>12.2</td>
</tr>
<tr>
<td>Tari(T)</td>
<td>13.5</td>
</tr>
<tr>
<td>Vanneque(Van)</td>
<td>11.95</td>
</tr>
<tr>
<td>Villaine(Vil)</td>
<td>11</td>
</tr>
<tr>
<td>Vrinat(Vri)</td>
<td>11.7</td>
</tr>
</tbody>
</table>

Table 8. Average score by Wine

This was partially due to the vastly different scales used by the raters. Note that the top ranked

Table 9. Average score by Wine
wine for Dovaz was rated a 15, which places it as the third highest ranked wine for Borejoux. To account for the difference in scales used by the individuals, we use binary comparison evaluation function from Equation 3.0.3. This approach is score invariant and considers solely the rank position of each individual wine according to each rater. This is equivalent to converting our original data to rank data. Here we assume ties use the smaller value (e.g. 3,7,7,9 maps to 4,2,2,1).

4.1. **Hungarian algorithm.** The Hungarian algorithm constructs an \( n \times m \) matrix where the count of each entry represents the number of times that item received a particular label.

To get a better sense of the structure we plot the counts and use a Kernel Density estimator to visualize the data in Table 6.

The optimal permutation is presented in Table 12. Yellow denotes the permutation that the Hungarian algorithm selected and green denotes the historic 1976 ranking. We see that in every row, Hungarian algorithm has chosen the label with the maximal row value, with the exception of Clos Du Val, which was ranked slightly higher. The most interesting aspect of this assignment is that Chateau Montrose, ranked third in 1976, ranked in 1st position here. Chateau Montrose has the most agreed upon label with five of 11 raters it the best wine. In total the Hungarian algorithm permutation makes use of 29 votes of the 110 cast while the historic ranking only represents 20. For the Hungarian Algorithm, we consider the accuracy as the fraction of correctly accounted for labels over total which in this example is 29/110 or .2654.
Figure 6. Ranking distributions by wine.
harmonic flow which are the inconsistencies denoted as gradient or consistent rankings and the sum of the curl and harmonic flow which are the inconsistencies denoted as \( U \). HodgeRank produces a consistent matrix

\[
\begin{bmatrix}
\text{Haut-Brion (HB)} & \text{Mouton-Rothschild (MR)} & \text{Stag's Leap (SL)} & \text{Clos Du Val (CL)} & \text{Ridge Vineyards (RV)} & \text{Chateau Montrose (CM)} & \text{Heitz Wine (HC)} & \text{Leoville Las Cases (CL)} & \text{Mayacamas (MV)} & \text{Freemark Abbey (FA)}
\end{bmatrix}
\]

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
3 & 1 & 3 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
2 & 2 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \\
3 & 3 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
2 & 1 & 1 & 1 & 2 & 0 & 3 & 0 & 0 & 1 \\
1 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 2 & 1 & 2 & 2 & 2 & 0 \\
1 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 3 \\
1 & 0 & 2 & 0 & 1 & 0 & 1 & 2 & 2 & 2 \\
\end{array}
\]

Table 12

4.2. **HodgeRank.** The first step when moving from permutations to binary comparison algorithms is to construct the aggregated binary comparison matrix \( Y \) which we have done below. There are no binary comparisons in which all the users agree and only one where ten do; Chateau Mouton–Rothschild is better than Mayacamas Vineyards. There are only two cases of nine way agreement: Stag’s Leap is better than Chateau Leoville Las Cases and Chateau Montrose is better than Chateau Leoville Las Cases.

\[
Y = \begin{bmatrix}
HB & MR & SL & CD & RV & CM & HC & CL & MV & FA \\
0 & 0 & 0 & -6 & -2 & 2 & -4 & -4 & -5 & -4 \\
0 & 0 & 2 & -7 & -5 & 1 & -7 & -5 & -7 & -7 \\
0 & -2 & 0 & -6 & -4 & 2 & -5 & -9 & -10 & -5 \\
6 & 7 & 6 & 0 & 5 & 6 & 2 & 3 & -1 & 3 \\
2 & 5 & 4 & -5 & 0 & 0 & -7 & -2 & -1 & -7 \\
-2 & -1 & -2 & -6 & 0 & 0 & -5 & -9 & -8 & -6 \\
4 & 7 & 5 & -2 & 7 & 5 & 0 & -1 & -1 & -3 \\
4 & 7 & 9 & -3 & 2 & 9 & 1 & 0 & -2 & -3 \\
5 & 5 & 10 & 1 & 1 & 8 & 1 & 2 & 0 & -1 \\
4 & 7 & 5 & -3 & 7 & 6 & 3 & 3 & 1 & 0 \\
\end{bmatrix}
\]

The objective of HodgeRank is to separate \( Y \) into three orthogonal subspaces: gradient, harmonic, and curl flow. We will consider only the gradient or consistent rankings and the sum of the curl and harmonic flow which are the inconsistencies denoted as \( U \). HodgeRank produces a consistent matrix

\[
C = \begin{bmatrix}
HB & MR & SL & CD & RV & CM & HC & CL & MV & FA \\
0 & 1.2 & -1.6 & 6.2 & 1.2 & -1.6 & 4.4 & 4.7 & 5.5 & 5.6 \\
1.2 & 0 & -4 & 7.2 & 2.4 & -4 & 5.6 & 5.9 & 6.7 & 6.8 \\
1.6 & .4 & 0 & 7.6 & 2.8 & 0 & 6.3 & 7.1 & 7.2 & 7.2 \\
-6.2 & -7.2 & -7.6 & 0 & -4.8 & -7.6 & -1.6 & -1.3 & -5.4 & -4 \\
-1.2 & -2.4 & -2.8 & 4.8 & 0 & -2.8 & 3.2 & 3.5 & 4.3 & 4.4 \\
1.6 & .4 & -. & 7.6 & 2.8 & 0 & 6.3 & 7.1 & 7.2 & 7.2 \\
-4.4 & -5.6 & -6.2 & 1.6 & -3.2 & -6 & 0 & 3 & 1.1 & 1.2 \\
-4.7 & -5.9 & -6.3 & 1.3 & -3.5 & -6.3 & -3 & 0 & .8 & .9 \\
-5.5 & -6.7 & -7.1 & .5 & -4.3 & -7.1 & -1.1 & -1.8 & 0 & .1 \\
-5.6 & -6.8 & -7.2 & .4 & -4.4 & -7.2 & -1.2 & -9 & -1 & 0 \\
\end{bmatrix}
\]
and an inconsistent matrix

\[
U = \begin{bmatrix}
    HB & MR & SL & CD & RV & CM & HC & CL & MV & FA \\
    HB & 0 & -1.6 & 0 & -0.8 & .4 & .4 & .7 & .5 & 1.6 \\
    MR & 1.2 & 0 & 1.6 & .2 & -2.6 & .6 & -1.4 & -1.1 & 1.7 & -2.2 \\
    SL & 1.6 & -1.6 & 0 & 1.6 & -1.2 & 2.1 & -2.7 & -2.9 & 2.2 \\
    CD & 0 & -2 & -1.6 & 0 & .2 & -1.6 & .4 & 1.7 & -1.5 & 2.6 \\
    RV & 0.8 & 2.6 & 1.2 & -2 & 0 & -2.8 & -3.8 & 1.5 & 3.3 & -2.6 \\
    CM & -0.4 & -0.6 & -2.1 & 1.6 & 2.8 & 0 & 1 & -2.7 & -0.9 & 1.2 \\
    HC & -0.4 & 1.4 & -1 & -0.4 & 3.8 & -1 & 0 & -0.7 & 0.1 & -1.8 \\
    CL & -0.7 & 1.1 & 2.7 & -1.7 & -1.5 & 2.7 & 0.7 & 0 & -1.2 & -2.1 \\
    MV & -0.5 & -1.7 & 2.9 & 1.5 & -3.3 & 0.9 & -0.1 & 1.2 & 0 & -0.9 \\
    FA & -1.6 & 0.2 & -2.2 & -2.6 & 2.6 & -1.2 & 1.8 & 2.1 & 0 & 0.9
\end{bmatrix}
\]

These are hard graphs to read, so to visualize this data more productively we have attached Figures 7, 8, showing the aggregate matrix and the inconsistencies removed respectively.

![Figure 7. Aggregate matrix](image-url)
Another interesting way of getting a sense of the distribution on our graphs is to look at the amount of intransitivity in each item. Here we have ordered the items from the least intransitive to the most and plotted a heat map of the results in Figure 9.

From the ordering we can see that the items with the most intransitivities removed are some of the most consistent highly ranked items we consider. Later we will see it is interesting to compare the structure of the heat maps as a way of interpreting if the inconsistencies are localized or general.

4.3. **AHP.** Using a exponentiated version of $\bar{Y}$ in section 4.2, we calculate the largest eigenvalue, and find its corresponding eigenvector to be used for ranking.

The largest eigenvalue in $\bar{Y}$ is:
Figure 9. Heat map of intransitivities in the 1976 Wine Ranking of the binary comparison matrix.

<table>
<thead>
<tr>
<th>Wines</th>
<th>HB</th>
<th>MR</th>
<th>SL</th>
<th>CD</th>
<th>RV</th>
<th>CM</th>
<th>HC</th>
<th>LC</th>
<th>MV</th>
<th>FA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
<td>-0.086</td>
<td>-0.026</td>
<td>-0.034</td>
<td>-0.615</td>
<td>-0.0737</td>
<td>-0.0038</td>
<td>-0.269</td>
<td>-0.215</td>
<td>-0.394</td>
<td>-0.589</td>
</tr>
<tr>
<td>Rank</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>10</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>6</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 14

$$\lambda_{\text{max}} = 33.9823879$$

and its correspond eigenvector and appropriate ranking is:

The consistency index of the matrix:

$$CI = \lambda_{\text{max}} - n(n - 1) = 2.665$$

Next AHP dictates that one calculate the consistency ratio. Using Saaty’s random results ($RI$) as a guide,

$$CR = CI / RI = 1.581$$

According to AHP, whenever $CR > .1$, one should consider asking the individuals to reconsider their evaluations.
4.4. Results. The results of the three different algorithms show reasonable variability given that original distributions.

Only the Hungarian algorithm placed an item to a rank more than ±2 positions from another. This kind of extreme rank position can be expected given the random assignment provided in the case of a tie. Ultimately the 1976 ranking was about establishing the supremacy of French wines over American ones, while in fact it proved the inverse. While AHP’s methodology places Stag’s Leap a second to Mouton–Rothschild, and HodgeRank indicates a tie with Chateau Montrose, this still serves to support the fundamental surprise result that American wines were achieving world class status.

5. College Ranking

The rankings of universities and colleges started as a relatively simple affair with Cattell in 1910, Babcock in 1911 and Hughes in 1925. Cattell and Hughes ranked the top twenty schools in Table 18, while Babcock clustered the universities into four tiers. Periodically the United States government considers establishing or enforcing a national system as Webster explores in BLANK. Today college rankings are exemplified by the US News and World Reports College Rankings, but there are many other organizations that produce world rankings or only within certain regions. Each uses their own proprietary algorithm and the diversity of ranking metrics and the multitude of sources of information often provide conflicting viewpoints. There is a great deal of disagreement regarding what these differing result indicate [21]. To our knowledge no one has yet tried to produce a comprehensive aggregate ranking from these disparate sources. We consider the following 38 years of journals and publications rankings.

<table>
<thead>
<tr>
<th>Wine</th>
<th>1976</th>
<th>HA</th>
<th>HR</th>
<th>AHP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accuracy</td>
<td>.26</td>
<td>.73</td>
<td>1.58</td>
<td></td>
</tr>
<tr>
<td>Stag’s Leap</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Mouton–Rothschild</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Chateau Montrose</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Haut–Brion</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Ridge Vineyards</td>
<td>5</td>
<td>7</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Leoville Las Cas</td>
<td>6</td>
<td>8</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>Heitz Wine</td>
<td>7</td>
<td>5</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>Clos Du Val</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Mayacamas</td>
<td>9</td>
<td>10</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Freemark Abbey</td>
<td>10</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 15

<table>
<thead>
<tr>
<th>Journal</th>
<th>Years</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>US News</td>
<td>2003–10</td>
<td>8</td>
</tr>
<tr>
<td>Forbes</td>
<td>2008–13</td>
<td>6</td>
</tr>
<tr>
<td>THE</td>
<td>2011–14</td>
<td>4</td>
</tr>
<tr>
<td>QS</td>
<td>2004–13</td>
<td>10</td>
</tr>
<tr>
<td>ARWU</td>
<td>2003–12</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 16
Each of the different rankings provides a slightly different approach to ranking. Our focus is simply to consider each periodical’s ranking rather than the methodology used to achieve this result.

5.0.1. U.S. News & World Report (US News). US News first published the America’s Best Colleges list in 1983 and began its yearly ranking in 1985. It represents the most well respected ranking in the United States, and is highly influential overseas. We consider the years from 2003-10 and the top 100 Universities rankings. Zeros indicate that the colleges ranking was not present in that years score. There are 102 universities included in this list.

5.0.2. Forbes. In 2008 Forbes began publishing the Center for College Affordability and Productivity list of American Colleges. They use 12 factors grouped in 5 broad categories each weighted individually. We consider the top 100 college rankings spanning the years 2008-2013. The rankings were very volatile at the bottom of our cut off. In order to have an irreducible matrix 0’s were replaced with 101, to denote that the university was above outside the 1-100 positioning. There were 169 universities included in this list.

5.0.3. Times Higher Education World Reputation Rankings (THE). The THE ranking were started in 2011 and reflect Thomson Reuters evaluations of the universities according to 13 performance parameters. We aggregated the data from 2011-2014 recording the top 100 rankings. Zeros indicate that there was no ranking that year. There were 118 colleges included in the ranking.

5.0.4. Quacquarelli Symonds (QS). Started in 2004, the Quacquarelli Symonds rankings represent a world wide ranking of the top 800 universities. We have the complete data for the top 100 universities from 2004-2013. There was a high degree of volatility within the rankings so zeros were replaced with 101 to denote that the ranking was outside the range. There were 160 universities across all years included in the list.

5.0.5. Academic Ranking of World Universities (ARWU). Started in 2003, ARWU is produced by Center for World-Class Universities at Shanghai Jiao Tong University. They use six indicators including number Noble Prizes/Fields Medals, number of Nature/Science publications, citation prominence from Thompson Scientific, per capita performances, number of articles in Science Citation Index and Social Science Citation Index. The top 500 universities are published. We consider the top 100 rankings, and the years 2003-2012. Zeros denote outside the range considered. There are 121 universities included in this ranking.

5.0.6. Year 2010. Finally, we consider the ranking for the year 2010 for each of the above rankings to evaluate how consistently rankings reflect across rankings. Each of the publications is treated as an individual rater and aggregated to reflect the ranking for a single particular year.

\[
\begin{array}{|c|c|c|}
\hline
\text{Algorithm} & \text{HA} & \text{HR} & \text{AHP} \\
\hline
\text{USNews} & .71 & .61 & 3.3 \\
\text{QS} & .86 & .71 & 8 \\
\text{ARWU} & .78 & .60 & .1 \\
\text{THE} & .87 & .7 & 1.98 \\
\text{Forbes} & .67 & .59 & 1.34 \\
\text{Year 2010} & .89 & .71 & 1 \\
\hline
\end{array}
\]

Table 17
5.1. **Results.** The results of the ranking are as follows:

We see the accuracy as calculated for each of the datasets below. Note that Forbes and then USNews report has the least consistency across all three algorithms. Next let us at the aggregation of top rates schools ranked by multiple rankings.
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Harvard</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
<tr>
<td>Princeton</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
<tr>
<td>MIT</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
<tr>
<td>Yale</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
<tr>
<td>Columbia</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
<tr>
<td>Stanford</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
<tr>
<td>Cal Tech</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
<tr>
<td>Brown</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
<tr>
<td>Duke</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
<tr>
<td>Oxford University</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
<tr>
<td>New York University</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
<tr>
<td>Imperial College</td>
<td>1.98</td>
<td>1.28</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
<td>4.22</td>
</tr>
</tbody>
</table>

Accuracy: Harvard 1.98, Princeton 1.98, MIT 1.98, Yale 1.98, Columbia 1.98, Stanford 1.98, Cal Tech 1.98, Brown 1.98, Duke 1.98, Oxford University 1.98, New York University 1.98, Imperial College 1.98, US Air Force Aca. 1.98, Uni. Cambridge 1.98.
<table>
<thead>
<tr>
<th>Colleges</th>
<th>index</th>
<th>Colleges</th>
<th>index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Northeastern Uni.</td>
<td>1</td>
<td>Yale</td>
<td>93</td>
</tr>
<tr>
<td>SUNY Col. Env. Sci. &amp; Forest</td>
<td>2</td>
<td>Harvard</td>
<td>94</td>
</tr>
<tr>
<td>Brigham Young</td>
<td>3</td>
<td>Princeton</td>
<td>95</td>
</tr>
<tr>
<td>Univ. of Missouri-Columbia</td>
<td>4</td>
<td>Uni. Cal - Berkeley</td>
<td>96</td>
</tr>
<tr>
<td>Uni. Alabama</td>
<td>5</td>
<td>Rice</td>
<td>97</td>
</tr>
<tr>
<td>Clemson</td>
<td>6</td>
<td>Dartmouth</td>
<td>98</td>
</tr>
<tr>
<td>Fordham University</td>
<td>7</td>
<td>CMU</td>
<td>99</td>
</tr>
<tr>
<td>Uni. Cal - Santa Cruz</td>
<td>8</td>
<td>Washington University in St. Louis</td>
<td>100</td>
</tr>
<tr>
<td>WPI</td>
<td>9</td>
<td>Brown</td>
<td>101</td>
</tr>
<tr>
<td>NC - Raleigh</td>
<td>10</td>
<td>Northwestern</td>
<td>102</td>
</tr>
</tbody>
</table>

Table 19. ranking from least to most inconsistent items.

Perhaps the most interesting result is the strange placement of Marquette University at position 16 in HA. While it regularly scores in the mind 90’s, there those scored 16 were placed elsewhere. A quick look at the data will show how this turned out.
Figure 10. Histograms of consistency
### 6. Top 35 in 2010

<table>
<thead>
<tr>
<th>Institution</th>
<th>2010</th>
<th>2009</th>
<th>2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boston College</td>
<td>27</td>
<td>39</td>
<td>40</td>
</tr>
<tr>
<td>Boston University</td>
<td>77</td>
<td>32</td>
<td>31</td>
</tr>
<tr>
<td>Brown</td>
<td>65</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>CMU</td>
<td>58</td>
<td>21</td>
<td>23</td>
</tr>
<tr>
<td>Cal Tech</td>
<td>6</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>College of William and Mary</td>
<td>31</td>
<td>49</td>
<td>51</td>
</tr>
<tr>
<td>Columbia</td>
<td>8</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>Cornell</td>
<td>70</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>Dartmouth</td>
<td>30</td>
<td>25</td>
<td>24</td>
</tr>
<tr>
<td>Duke</td>
<td>14</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td>Emory</td>
<td>7</td>
<td>48</td>
<td>41</td>
</tr>
<tr>
<td>Georgetown</td>
<td>65</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>Harvard</td>
<td>8</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>Hong Kong University</td>
<td>23</td>
<td>98</td>
<td>100</td>
</tr>
<tr>
<td>Imperial College London</td>
<td>7</td>
<td>34</td>
<td>34</td>
</tr>
<tr>
<td>Johns Hopkins</td>
<td>18</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>Kyoto University</td>
<td>25</td>
<td>38</td>
<td>37</td>
</tr>
<tr>
<td>MIT</td>
<td>10</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>NYU</td>
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<tr>
<td>Northwestern</td>
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<tr>
<td>Notre Dame</td>
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<tr>
<td>Oxford University</td>
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<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Penn State</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Princeton</td>
<td>17</td>
<td>26</td>
<td>26</td>
</tr>
<tr>
<td>Rice</td>
<td>13</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Stanford</td>
<td>34</td>
<td>41</td>
<td>43</td>
</tr>
<tr>
<td>Tufts</td>
<td>4</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>U Penn</td>
<td>12</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
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**Table 21. US News Rank Aggregation results**

7. Conclusion

The rank aggregation tools we consider here are coarse computational tools that rely on strong assumptions about optimality. Despite the inherent drawbacks from such an approach, they serve as dependable and computationally tractable tools employable on a broad range of data sets. In summary, we considered three algorithms, Hungarian algorithm, HodgeRank, and Analytic Hierarchy protocol and demonstrate their comparative advantages and disadvantages on multiple datasets. Most importantly we collected and analyzed over 40 years of college ranking data in a reproducible format and compared and contrasted the results of the algorithms accordingly.

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Table 22. QS Rank Aggregation


