Some Problems in Proof Complexity and Combinatorial Optimization

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Abstract

Lovász and Schrijver [34] introduced several lift and project methods for 0-1 integer programs, now collectively known as Lovász-Schrijver (LS) hierarchies. Several lower bounds have since been proven for the rank of various linear programming relaxations in the LS hierarchy. In this paper we study lower bounds on the rank in the more general $LS_*$ hierarchy, which was first investigated by Grigoriev et al [26]. In particular we show that the PHP inequalities have a $\log_2 n$ rank refutation in $LS_*$ and we prove that this is tight. We also extend the strong rank lower bounds for MAX-CUT, Sparsest Cut etc, studied in Charikar et al. [12], for Sherali-Adams to weaker rank lower bounds in $LS_*$ hierarchy.
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Chapter 1

Introduction and Background

1.1 Introduction

In the past two decades many linear programming relaxation methods have been investigated due to their potential in various optimization problems [14]. Thus far most of the results have been negative. In this paper we prove some simple rank lower bounds for a linear programming hierarchy originally defined by Lovász and Schrijver [34] and also studied by Grigoriev et al. [26].

The paper is organized as follows. In the first chapter we give a basic refresher to definitions in convex optimization and proof complexity. We then introduce \( LS^*_\) relaxations - the main object of investigation. In the second chapter we prove various bounds on \( PHP \), \( MAX-CUT \) and other inequalities in the corresponding sections. Finally, we mention some open problems. Some of the proofs have been shifted to the appendix.

1.2 Preliminaries and definitions

In this section a basic refresher to convex optimization, lift and project methods and proof complexity is given, more details of which can be found in the books and surveys [32, 14, 43] and the references therein.

1.2.1 Basic facts about convex optimization

A polyhedron in \( \mathbb{R}^n \) is the set of solutions of a system of linear inequalities. A bounded polyhedron is called a polytope. Any face of a polyhedron \( \{ x \in \mathbb{R}^n | Ax \geq b \} \) can be written in the form \( \{ x \in \mathbb{R}^n | A_1 x = b, A_2 x \geq b \} \) for some partition \( A_1, A_2 \) of the matrix \( A \). A cone in \( \mathbb{R}^n \) is a set closed under addition and multiplication by positive scalars. A polyhedral cone is the set of solutions of a homogenous system of linear inequalities. Given a cone \( C \)
its polar (sometimes also called dual) $C^*$ is defined as $\{y | y^T x \geq 0, x \in C\}$. Intuitively, the coordinates of the polar $C^*$ represent the coefficients of the set of valid inequalities of $C$. An *extreme* vector in a cone is any vector that can not be expressed as a convex combination of other vectors. Below we give a short refresher on a couple of lemmas which are basic tools when thinking about lift and project methods.

The *Fourier-Motzkin elimination* [35] is an analogue of gaussian elimination but for inequalities. Many times (esp. when working with the algebraic definitions of lift and project) one is faced with a need to eliminate some variables - the Fourier-Motzkin elimination is helpful here. The idea is simple as the following example shows. Given inequalities

\[
a_1 x + a_2 y + z \geq a_3, b_1 x + b_2 y - z \geq b_3, c_1 x + c_2 y - z \geq c_3 ... a_i, b_i, c_i \in \mathbb{R}
\]

we can eliminate the variable $z$ (at the cost of increasing the number of inequalities) to obtain the system:

\[
\min(b_1 x + b_2 y - b_3, c_1 x + c_2 y - c_3) \leq z \leq \max(a_1 x + a_2 y - a_3).
\]

Next we pairwise compare the lower and upper bounds in the second system of inequalities above. The claim is that the second system has a solution if and only if the first has a solution. Furthermore the second system (after eliminating $z$) contains all the extreme inequalities in $x,y$ implied by the first. The generalization to $n$ variables is now straightforward and the proof can be found in [35].

Finally, we state the Farkas Lemma (an important rule to characterize when a system of inequalities has a positive solution).

**Lemma 1.2.1** ([35]). Given $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ then exactly one of the following occurs:

1. $\exists x \in \mathbb{R}^n$ such that $Ax = b$ and $\forall i x_i \geq 0$.
2. $\exists y \in \mathbb{R}^m$ such that $y^TA \geq 0$ and $y^Tb < 0$.

The lemma is equivalent to the following basic statement in proof complexity: If a given system of inequalities is inconsistent then one can derive the inequality $-1 \geq 0$ from it by taking positive linear combination of the given inequalities. The lemma can be used to prove an equivalent algebraic characterization of Lovász-Schrijver hierarchy from the geometric definition (next subsection). The proofs can be found in [35].

### 1.2.2 Lift and project methods

Methods from non-linear algebra (including Gomory-Chvátal cutting planes, Lovász-Schrijver lift and project etc) have been studied for a while. The
main motivation seems to start from integer programming problems. The idea being that a small set of well defined operations on the facets of a given polytope would eventually eliminate all non-integer vertices of the polytope and leave one with the integer polytope. Note that all through this article we will only deal with 0-1 integer programming problems. In this subsection we only give a very basic refresher of Lovász-Schrijver [34] methods in this regard. The reader is referred to the surveys [14, 32] for details about this and other related methods.

The concept of a ‘lift’ (not necessarily a Lovász-Schrijver lift) is informally illustrated by an example below. Note that by ‘lifting’ a linear program one can increase the number of variables (say by a polynomial factor) and obtain a potentially simple new linear program with (greatly) reduced number of facets (i.e. constraints). The example below is from Lovász [33]. Given a comparability graph \( G \) (i.e. the graph \( (V, \leq) \) defined from a partial order on the set \( V \)) the stable set polytope is described by the constraints

\[
0 \leq x_i \leq 1 \quad \forall i \in V, \\
\sum_{i \in C} x_i \leq 1 \quad \forall C \in \text{Chain}(V).
\]

Since there are exponentially many chains, the polytope above has an exponential number of facets. However, the following ‘lifted’ linear program has just \( O(|V|^2) \) facets:

\[
0 \leq x_i \leq y_i \leq 1 \quad \forall i \in V, \\
y_i + x_j \leq y_j \quad \forall i, j \in V, \ i < j.
\]

Furthermore by ‘projecting’ the above polytope on \( x \) variables (say by Fourier-Motzkin) one gets the stable set polytope for comparability graphs as above.

Next we restate the (geometric) definition of Lovász-Schrijver (LS) lift and project. Given polyhedon \( K := \{ x \in \mathbb{R}^n | Ax \geq b \} \) define the homogenization of \( K \) to be the cone \( \tilde{K} := \{ \lambda (1, x) | x \in K, \lambda \geq 0 \} \). In the following definition homogenization is ensured by introducing an extra variable \( x_0 \) and changing the constraints \( Ax \geq b \) and \( x_i \in [0, 1] \) to \( Ax \geq b x_0 \) and \( x_i \in [0, x_0] \) respectively. The polytope corresponding to the homogenized cone \( \tilde{K} \) is obtained by intersecting it with the hyperplane \( x_0 = 1 \). Denote by \( Q_n \) the cube \([0, 1]^n\) for some \( n \in \mathbb{N} \).

**Definition 1.2.2 ([34]).** Given convex cones \( \tilde{K}_1, \tilde{K}_2 \) in \( \mathbb{R}^{n+1} \) define the cone \( M(\tilde{K}_1, \tilde{K}_2) \) (the lifted cone) as the cone consisting of all \( (n + 1) \times (n + 1) \) matrices \( Y \) in \( \mathbb{R} \) satisfying the conditions:

1. \( Y \) is symmetric
2. \( Y_{ii} = Y_{i0} \)
3. $\tilde{K}_1^* Y \tilde{K}_2^* \geq 0$.

Let $N(\tilde{K}_1, \tilde{K}_2)$ denote the projection $Y_0$ of $M(\tilde{K}_1, \tilde{K}_2)$.

Note that $N(\tilde{K}, \tilde{Q}_n)$ refers to the cone obtained after a single Lovász-Schrijver lift and project step. Let $N(K, Q)$ or simply $N(K)$ denote the corresponding polytope. Let $N^r(K)$ denote $N$ applied $r$ times in succession to $K$. If $N^r(K)$ is the integer polytope then $r$ is also known as the rank of $K$. Since it is easy to go between polytopes and cones, we will usually drop the $\sim$ and accompanying extra notation unless we want to emphasize it. The intuition behind the use of the terms ‘lift’ and ‘project’ in context of definition 1.2.2 will become clearer towards the end of the next subsection.

Lovász and Schrijver [34] also prove that after $n$ rounds of applying the operator $N$, any $n$ dimensional polytope converges to its integer polytope. They also showed that it is possible to efficiently optimize over $N^r(K, Q)$ and the semidefinite version $N^r_+(K, Q)$. The algorithm essentially constructs separation oracles for $N(K, Q)$ and $N_+(K, Q)$ from that of $K$. Note that the Sherali-Adams (SA) hierarchy [48] is just like the LS hierarchy of lift and project but with one difference: the project step (i.e. projection back to the initial variables) is applied only at the end of all the lift steps. For more details about Sherali-Adams refer to the survey [32] or the paper [12].

### 1.2.3 Basic facts about propositional proof complexity

As explained later in this subsection Lovász-Schrijver and other similar procedures can also be thought of as proof systems [38] and are also studied in proof complexity literature. Proof complexity is a very large area and the following is a very brief refresher. More details are available in various books and surveys [30, 47].

Given a decision version of any combinatorial optimization problem $O$ it naturally defines a language $L \subseteq \{0, 1\}^*$. A proof system [17] for a language $L$ is a polytime computable function $P : \{0, 1\}^* \rightarrow L$ satisfying:

1. Completeness: $x \in L \implies \exists y \, P(y) = x$,
2. Soundness: $\exists y \, P(y) = x \implies x \in L$.

Let $TAUT$ denote the language of tautologies in some propositional logic. A proof system for $TAUT$ is a propositional proof system. A proof system $V$ is polynomially bounded if $(\exists k)(\forall x)(P(y) = x \implies ((\exists z)(P(z) = x \land |z| \leq |x|^k)))$. For $P(z) = x$ the length of string $z$ denoted $|z|$ above is called the size of the $P$-proof of $x$. Cook and Reckhow [17] showed that there exists a polynomially bounded propositional proof system if and only if $NP = coNP$. Since $NP \neq coNP \implies P \neq NP$, a natural way to make progress towards proving $P \neq NP$ is to prove lower bounds for a family of proof systems where each successive proof system is able to polynomially...
simulate its predecessors. The final aim then being to prove that some
Extended Frege system is not polynomially bounded.

Besides the size of the proof there exist other complexity measures which
may not make sense for a general proof system but do make sense for proof
systems with additional structure. A Frege system $F$ is a sound and impli-
cationally complete propositional proof system having finitely many axiom
schemes and inference rules of the form $\frac{A_1, \ldots, A_k}{A_0}$, where $A_i$ are propositional
formulae. $F$ is sound if every $F$ provable formula is a tautology. An $F$-proof
of $\phi$ is a sequence of formulae $\phi_1, \ldots, \phi_j$ ending with $\phi$ such that each $\phi_j$ is
either an axiom or obtained from previous $\phi_i$ by substitutions into inference
rules of $F$. $F$ is implicationally complete if there exists an $F$-proof of $\phi_0$
from $\phi_1, \ldots, \phi_k$ whenever $\phi_1, \ldots, \phi_k$ semantically imply $\phi_0$. Frege proof sys-
tems are important in mathematical logic, and proof complexity but we do
not discuss anything more about them here since we are mainly interested
in weaker proof systems. The reader is referred to the book [30] for details.
Note that any Frege proof has a natural directed acyclic graph structure.
In fact any proof in a propositional proof system with inference rules re-
sembling Frege can be naturally viewed as a directed acyclic graph where
each node represents an intermediate formula in the proof. The size of the
proof is then the number of nodes in the graph and the depth is the longest
path from the root to a leaf. Proof depth (more commonly called rank since
depth is also used to denote the maximum formula depth) has been studied
more commonly for integer linear programming based proof systems like
$CP, LS$ etc. In this paper we will be concerned with bounds on rank. Note
that a size lower bound implies a corresponding logarithmic lower bound on
rank.

In the case of proof systems like Resolution, CP, LS etc where the proof
lines are not expressive enough to encode arbitrary propositional formulae
we use the fact (akin to Cook-Levin reduction [43]) that any propositional
formula can be encoded as a CNF. For weak proof systems like Resolution,
CP, LS etc the given tautology is negated and expressed in CNF or some
similar suitable encoding (like linear inequalities) depending on the proof
system. A proof is then a valid refutation ending with the empty clause
or the empty polytope. Proof complexity lower bounds for $CP, LS$ aim to
show that even in this special case the rank is large.

Resolution is one of the simplest propositional proof systems. It is sound
and complete and resembles Frege in structure. Any proof line is just a
CNF clause. It is known that $PHP^{n+1}_n$ requires exponential size resolution
proofs [28]. Similarly it is also known that a random $k$-SAT formula requires
exponential size refutation in resolution [7]. Resolution was generalized to
$Res[k]$ [31] where the proof lines are $k$-DNF formulae and it is known that
$PHP^{n+1}_n$ requires exponential size refutation in $Res[k]$ [6] (and so does any
bounded depth Frege system [6]). Several algebraic proof systems have also
been defined and these are of more interest to us because of the similarity of
their inference rules to $LS$ type proof systems. The main idea behind these proof systems is to encode the propositional axioms as polynomials in the Smolensky ring $S_n(\mathbb{F}) = \mathbb{F}[[x_i : i \in [n]]]/(\{x_i^2 - x_i : i \in [n]\})$ so that a refutation corresponds to some algebraic property of the polynomials (say the non-existence of common root). Examples of such proof systems include (in increasing order of strength): the Nullstellensatz [5], Polynomial Calculus ($PC$) [16], Polynomial Calculus with Resolution ($PCR$) [2], $PCR[k]$ [24], and multilinear proofs [40]. Various size, and degree lower bounds are known for all of them (except multilinear proofs). We only restate the definition of the last proof system here i.e. multilinear proofs, since it is fairly recent.

Let $C$ be the initial set of polynomials over variables $\{x_1, ..., x_n, x'_1, ..., x'_n\}$ obtained by encoding the given CNF formulae. The aim is to prove insolvability of the equations in $C$. The proof also uses $x_i x'_i = 0$ and $x_i + x'_i = 1$ as axioms. The lines of the proof are multilinear formulae. The proof uses the following inference rules:

1. $\frac{p=0 \ q=0}{\alpha p + \beta q = 0}$ for scalars $\alpha, \beta$.
2. $\frac{p=0}{p q = 0}$ for any multilinear polynomial $p \cdot q$.

The proof terminates with the unsatisfiable equation $1 = 0$.

Next we give a characterization of $LS$ as an algebraic proof system. Given a CNF $(x_1 \lor \neg x_2) \land (x_1 \lor \neg x_3)$ we can encode it by the inequalities $x_1 + 1 - x_2 \geq 1$ and $x_1 + 1 - x_3 \geq 1$ i.e. the inequalities have a 0-1 solution if and only if the CNF is satisfiable. Next we relax the variables $x_1, x_2$ to be in $[0, 1]$ so that the integer polytope defined by the above inequalities is empty if and only if the original propositional formula was unsatisfiable. Since $n$ rounds of Lovász-Schrijver converges to the 0-1 polytope applying 3 rounds of Lovász-Schrijver to the above pair of inequalities will yield an empty polytope if and only if the original CNF is unsatisfiable. This naturally leads to a view of $LS$ as a proof system. Using essentially the Farkas Lemma and some linear algebra the $LS$ lift and project (1.2.2) can be alternatively characterized as follows:

**Lemma 1.2.3** [20, 34]. Given an initial polytope $K$ in $\mathbb{R}^n$, point $x \in N(K)$ iff it satisfies all linear inequalities of the form

$$\sum_{i,j} \mu_{i,j} x_i h_j(x) + \sum_{i,j} \nu_i(1-x_i) h_i(x) + \sum_i \gamma_i (x_i^2 - x_i)$$

where $h_i$ are the constraints of $K$, $\mu_i, \nu_i$ are non-negative reals and $\gamma$ is any real.

Therefore one can formally define the $LS$ proof system as follows.
Definition 1.2.4. Given a set \( C \) of linear inequalities on the variables \( \{x_1, \ldots, x_n\} \) and add to that axioms \( x_i^2 - x_i = 0 \), we have the following inference rules for LS:

1. \( \frac{p \geq 0}{pq \geq 0} \) where \( \deg(pq) \leq 2 \) and \( q \in \{x_i, 1 - x_i : i \in [n]\} \).

2. \( \frac{p \geq 0, q \geq 0}{\alpha p + \beta q \geq 0} \) for \( \alpha, \beta \in \mathbb{R}^+ \).

A valid refutation must obtain the contradiction \(-1 \geq 0\).

Observe the resemblance of LS inference rules to those of PCR / multilinear proofs. The LS rank of a refutation ignores any contribution due to the second inference rule in definition 1.2.4 and only concentrates on the number of lift and project steps among paths. Therefore in view of Lemma 1.2.3 the rank as defined from a propositional proof system perspective coincides with the rank based on the geometric definition 1.2.2.

The Sherali-Adams (SA) hierarchy has been compared to LS hierarchy in detail by Laurent [32]. As explained earlier we encode all our axioms by linear inequalities as in [12, 19] so we can use a definition of Sherali-Adams hierarchy restricted to linear inequalities as given below\(^4\). The definition below is adapted from that in section 3.2 in [32].

Definition 1.2.5. Given a set \( C \) of linear inequalities on the variables \( \{x_1, \ldots, x_n\} \) in \( S_n(\mathbb{R}) \), we have the following inference rule for \( r \) rounds of Sherali-Adams:

1. \( \frac{p \geq 0}{pq \geq 0} \) where \( \deg(pq) \leq r + 1 \) and \( q \in \{x_i, 1 - x_i : i \in [n]\} \).

2. \( \frac{p \geq 0, q \geq 0}{\alpha p + \beta q \geq 0} \) for \( \alpha, \beta \in \mathbb{R}^+ \).

Again a valid refutation must obtain \(-1 \geq 0\).

The intermediate formulae involved in deriving the empty polytope (both for SA and LS) are essentially multilinear polynomials (due to the axioms \( x_i^2 = x_i \)). The same formulae can be viewed as lifted inequalities by mapping monomials \( \Pi_{i \in I} x_i \) to lifted real valued variables \( x_I \) for \( I \subseteq [n] \). Therefore a multinomial \( \sum_j \alpha_j \Pi_{i \in I_j} x_i \) would map to the lifted inequality \( \sum_j \alpha_j x_{I_j} \geq 0 \). In case of LS the lifted variables correspond to subsets of \( [n] \leq 2 \), while in case of \( r \) rounds of SA they correspond to subsets of \( [n] \leq r+1 \). The intuition behind Lemma 1.2.3 for Lovász-Schrijver hierarchy extends to Sherali-Adams as well. The SA \( r \)-lifted polytope consists of the polytope defined by all lifted inequalities obtained within \( r \) rounds of SA and the corresponding SA projected polytope is the polytope defined by inequalities with monomials of

\(^4\)Note the similarity to \( LS^r \) studied by Grigoriev et al. [26].

\(^4\)In case of constant terms we can either choose to represent them by a lifted variable \( x_0 = 1 \) (i.e. homogenize the lifted polytope) or leave them as constants, it will not significantly change any arguments in the rest of the paper.
degree at most 1. Given an unsatisfiable propositional formula encoded as linear inequalities to prove a lower bound of $r$ on $SA$ (or $LS$) rank it suffices to show that the polytope defined by the lifted inequalities after $r$ rounds of $SA$ is non-empty. The reader is referred to the surveys [14, 32] or the papers [12, 34] for more discussions about lifted inequalities and corresponding lifted polytopes.

Finally, the $LS_*$ proof system was defined by Grigoriev et al [26] removing the constraint $q \in \{x_i, 1 - x_i\}$ in definition 1.2.4.

**Definition 1.2.6.** ([26]) Given a set $C$ of linear inequalities on the variables $\{x_1, \ldots, x_n\}$ and add to that axioms $x_i^2 - x_i = 0$, we have the following inference rules for $LS_*$:

1. $p \geq 0, q \geq 0, p \cdot q \geq 0$ where $\deg(p \cdot q) \leq 2$.
2. $p \geq 0, q \geq 0, \alpha p + \beta q \geq 0$ for $\alpha, \beta \in \mathbb{R}^+.$

A valid refutation must obtain the contradiction $-1 \geq 0$.

For brevity we assume that the inequalities $x_i \geq 0$ and $1 - x_i \geq 0$ are implicitly present in $C$ unless mentioned otherwise. Again only use of inference rule 1 counts towards the $LS_*$ rank. Observe that the polytope defined by rank $r$ inequalities in $LS_*$ is a subset of the polytope defined by rank $r$ inequalities in $LS$. In this paper we will concentrate on lower bounds (mainly depth/rank) in the $LS_*$ proof system.

### 1.2.4 The $LS_*$ hierarchy

In this subsection we give some background about the $LS_*$ hierarchy which was first introduced in a geometric setting by Lovász and Schrijver [34] in their initial paper on lift and project methods in combinatorial optimization but not studied beyond one lift and project step (probably because of algorithmic reasons). It was introduced again by Grigoriev et al. in their initial paper [26] while studying lower bounds on such proof systems in an algebraic setting.

Lovász and Schrijver [34] also defined the operator $N(K, K)$ as in 1.2.2. Let $N^2(K, K)$ denote the polytope $N(N(K, K), N(K, K))$ and similarly one defines $N^r(K, K)$ as $N(N^{r-1}(K, K), N^{r-1}(K, K))$. Let $N^r_*(K)$ denote the polytope defined by inequalities of rank at most $r$ in $LS_*$ hierarchy. The following lemma relates the geometric [34] and algebraic [26] definitions (the proof is exactly similar to the corresponding equivalence for $LS$ and $LS_+$ in Lemma 1.2.3 with the cone $Q$ replaced by $K$, and is therefore omitted).

**Lemma 1.2.7.** $N^r(K, K) \equiv N^r_*(K)$.

Lovász and Schrijver [34] also construct a weak separation oracle for $N(K, Q)$ and $N(K, K)$ from a weak separation oracle for $K$. In the latter
case an assumption is made that the polytope $K$ is given as a polynomial size system of linear inequalities and so one iterates over the facets of $K$ in polytime to check the third condition in the definition 1.2.2. This assumption implies that one cannot perform algorithmic optimization over $r$ rounds of $LS_*$ in polynomial time by simply iterating over the facets of $N^{r-1}(K,K)$ as before, since the intermediate polytopes $N(K,K)$ (or $N^2(K,K) = N(N(K,K), N(K,K))$ etc) may have exponentially many facets in general $\text{iii.}$ It is possible that the situation for $LS_*$ may be similar to the case for Gomory-Chvátal cuts where the corresponding separation problem is known to be NPC [22]. The discussion above implies that the $LS_*$ proof system which has a geometric as well as algebraic characterization could be very strong (as compared to say $LS$ or $LS_+$) and it motivates us to investigate further upper and lower bounds for rank (and eventually size) for the $LS_*$ hierarchy. The following is a brief description of results already known about the $LS_*$ hierarchy.

Thus far no strong upper bounds seem to be proven specifically for $LS_*$. Although Lovász and Schrijver [34] define $N(K,K)$, [34] does not prove any specific bounds, and in the discussion following theorem 2.13 in [34] they leave open the question of comparing the strengths of $N(K,Q)$ and $N+(K,K)$ with $N(K,K)$.

**Definition 1.2.8.** ([26, 29]) Let $x_i \in \{0,1\}$ for $i \in [n]$ then symmetric knapsack refers to the following equality:

$$\sum_{i=1}^{n} x_i = r \in \mathbb{R}. \quad (1.2.1)$$

Let $SK_n$ denote the case when $r = \frac{n}{2}$ and $n$ odd.

Note that for odd $n$ $SK_n$ is unsatisfiable. Grigoriev et al. [26] prove the following lower bound for symmetric knapsack $\text{iv}$

**Theorem 1.2.9** ([26]). For odd $n$:

1. Any $LS_+$ refutation of $SK_n$ has rank at least $\frac{n}{2}$.  

$\text{iii}$ For example consider the natural independent/stable set linear program for a graph $G$ (the initial polytope is denoted by $FRAC(G)$) only the positivity and edge constraints are essential. Lovász and Schrijver [34] show that $N(FRAC(G),Q)$ consists of exactly the positivity constraints, edge constraints and the odd hole constraints. Now let $G$ be the graph formed by joining $2^{\frac{n-1}{3}}$ copies of $C_4$ end to end in a chain and adding an edge $e$ between the two extreme end vertices of the chain. The graph $G$ has odd cycles so $FRAC(G) \neq STAB(G)$, where $STAB(G)$ is the 0-1 stable set polytope for $G$. In fact all odd cycles have to use the edge $e$ therefore the graph is t-perfect [23, 49]. For t-perfect graphs Lovász and Schrijver [34] show that $STAB(G) = N(FRAC(G),Q)$ and so $STAB(G) = N(FRAC(G),FRAC(G))$. Note that $G$ has an exponential number of chordless odd cycles each of length $2^{\frac{n-1}{3}} + 1$ and each of those constraints is essential. Hence $G$ provides a pathological example where the projected cone has an exponential number of facets when compared to the lifted cone.

$\text{iv}$ They claim the bounds extend to Tseitin tautologies also.
2. Any $LS_{+,*}$ refutation of $SK_n$ has rank at least $\log_2 n - 1$.

The only known super-logarithmic rank lower bound in $LS_+$ is due to Beame et al. [4]. Define $\mathcal{GPHP}_m$ as just $PHP^m_{m+1}$ on a bigraph $(U, V)$ with $\deg(u) \leq 5$ for $u \in U$. Lemma 4.2 in Beame et al. [4] (restated below) proves a $n^{\Omega(\frac{1}{k})}$ refutation rank lower bound in a very strong proof system (i.e. $R^{cc}(k)$) for formula $G := Lift_{k-1}(\mathcal{GPHP})$ on $n$ variables with $k \leq (1 - \epsilon) \log \log n$.

**Lemma 1.2.10** ([4]). There is a family of bipartite graphs $\mathcal{G}$ and a family of polysize CNF formulae $G := Lift_{k-1}(\mathcal{GPHP})$ on $n$ variables that require refutation rank $n^{\Omega(1/k)}$ and tree-like size $\exp(n^{\Omega(1/k)})$ in any $R^{cc}(k)$ system for any $k \leq (1 - \epsilon) \log \log n$ for some positive absolute constant $\epsilon$.

Beame et al. [4] define $T^{cc}(k)$ as a proof system such that its proof lines consist of arbitrary boolean functions that can be evaluated by an efficient $k$-party randomized communication protocol. $T^{cc}(k)$ proofs include $Th(k)$ proofs i.e., proofs where the proof lines are degree $k$ polynomials, as a special case. $R^{cc}(k)$ proofs [4] are even more general and include $T^{cc}(k)$ proofs. The fact that these proofs have bounded fan-in is not a problem since Cartheodory’s theorem (about convex polytopes) shows that $LS^{k,v}_+ \ast$ proofs can be simulated by $Th(k + 1)$ proofs with only $O(\log n)$ factor increase in rank and polynomial increase in size [4]. The above summary together with Lemma 1.2.10 implies a $n^{\Omega(1/k)}$ rank lower bound for $Lift_{k+1}(\mathcal{GPHP})$ in $LS^{k,v}_+ \ast$, when $k \leq (1 - \epsilon) \log \log n$.

Finally, note that theorem 4.1 in Goemans et al. [25] (equivalently the general protection lemma i.e. Lemma 9 in [9]) does not seem to extend (at least in a straightforward manner) to $LS_+$, hence the linear rank lower bound proofs in [9] don’t extend to $LS_+$.
Chapter 2

Rank Bounds for $LS_*$

In this chapter we study rank (i.e. proof depth) lower bounds for several problems in the $LS_*$ hierarchy.

In the first section we prove a logarithmic upper and lower bound on the $LS_*$ rank of $PHP$ (i.e. Pigeon Hole Principle). The motivation behind studying $PHP$ comes from the fact that it is one of the cornerstone problems in proof complexity as witnessed by the survey [42]. The $PHP^{n+1}_n$ (falsely) claims that there exists a (possibly) multivalued everywhere defined injection between the two partitions of the bigraph $K_{n+1,n}$. A refutation of this claim in a proof system successfully derives a contradiction using the proof rules starting from the suitably encoded axioms for $PHP$. In the case of resolution $PHP$ can be encoded as the following set of propositional formulae.

**Definition 2.0.11.** [42] Define $PHP^m_n$ over $x_{ij} \in \{0, 1\}$ by

\[
Q_i := \bigvee_{j \in [n]} x_{ij} \quad (2.0.1)
\]

\[
Q_{jk,i} := \neg x_{ji} \lor \neg x_{ki} \quad (j, k \in [m], j \neq k, i \in [n]) \quad (2.0.2)
\]

It has been shown that any refutation of $PHP^{n+1}_n$ in Resolution requires $2^{\Omega(n)}$ size [28] (and therefore polynomial depth). The following encoding of $PHP$ has also been used to prove polynomial degree lower bounds for the Polynomial Calculus proof system [41, 29].

**Definition 2.0.12.** [41, 29] Define $PHP^m_n$ over $S_{mn}(\mathbb{R})$ by the polynomials

\[
Q_i := 1 - \sum_{j \in [n]} x_{ij} \quad (2.0.3)
\]

\[
Q_{jk,i} := x_{ji}x_{ki} \quad (j, k \in [m], j \neq k, i \in [n]) \quad (2.0.4)
\]
Buss [10] showed that \( PHP \) has polynomial size refutations in Frege systems and [6] show an exponential size lower bound in bounded depth Frege\(^1\). It is known that \( PHP \) has polynomial size and logarithmic rank in the Gomory-Chvátal cutting planes (\( CP \)) proof system [38]. However, results by Bonet, Pitassi and Raz [8] followed by the result of Pudlak [37] showed an exponential size lower bound in \( CP \). It is known (via Pudlak [38]) that \( PHP \) has a polynomial size upper bound for \( LS \) and Grigoriev et al. [26] proves a linear rank lower bound for \( PHP_{n+1} \) in \( LS \). The following section proves logarithmic rank upper and lower bounds for \( PHP \) in \( LS \) which motivates us to consider other candidate problems for proving a super-logarithmic rank lower bound in the \( LS \) hierarchy.

Lovász-Schrijver type methods have recently received much attention [3, 9, 1, 45, 46, 12, 44, 14] from the combinatorial optimization community especially since many known linear programming and semidefinite programming based approximation algorithms can be obtained using a constant number of rounds of \( LS \) or \( LS_\ast \) [14]. There are also some examples where the approximation ratio can be improved when one uses larger number of rounds of \( LS/LS_\ast \) [15, 14]. However, strong (linear) rank lower bounds are also known for many problems in the \( LS, SA, LS_\ast \) and even the Lasserre (in some cases) hierarchies. In the next section we show via a very simple argument that the method of Charikar et al. [12] extends to prove several logarithmic lower bounds in \( LS_\ast \) for well studied optimization problems like MAX-CUT, Sparsest Cut etc. Since \( LS_\ast \) is not known to have good algorithmic properties it is particularly interesting (at least to the author) to know whether any of these problems have a super-logarithmic rank in the \( LS_\ast \) hierarchy.

Finally, the chapter concludes with a comparison of \( LS_\ast \) rank with \( LS_\ast \) and \( SA \) rank. We show that there exist systems of inequalities with linear \( LS_\ast \) and \( SA \) rank but only constant \( LS_\ast \) rank.

### 2.1 PHP

In this section the aim is to prove that the \( LS_\ast \) rank of \( PHP \) (and some related problems) is \( \log_2 n \).

**Definition 2.1.1.** Define \( SPHP^n_1 \) polytope by the linear inequalities

\[
Q_{ij} := 1 - x_i - x_j \geq 0, \quad (\forall i \neq j, i, j \in [n]), \quad x_i \in [0,1].
\]  

(2.1.1)

The S here stands for satisfiable.

\( SPHP^n_1 \) was studied before by Pitassi and Segerlind [36] as an example to show that a rank-size relation of the form \( \text{rank} \leq \sqrt{n \log(\text{Size}_T)} \) does

\(^1\)In relation to bounded depth Frege, ‘depth’ signifies maximum formula depth in the proof lines and not proof depth.
not hold for \( LS \) in the case of arbitrary inequalities. \( SPHP^n_1 \) polytope corresponds to the matchings in the star \( K_{1,n} \). Proving upper bound on the rank of \( SPHP^n_1 \) will immediately imply the same for \( PHP^n+1 \).

**Theorem 2.1.2.** After \([\log_2 n] – 1\) rounds of \( LS \) the system of inequalities \( SPHP^n_1 \) converges to the integer polytope.

**Proof.** For simplicity assume \( n \) is a power of 2. Observe that one round of \( LS \) gives \( x_i(1 – x_i – x_j) \geq 0 \) (since \( x_i^2 – x_i = 0 \)) so that \( x_i x_j \leq 0 \) which implies one can effectively set \( x_i x_j = 0 \) (2.1.2) as \( x_i x_j \geq 0 \).

Now in one round of \( LS^* \) we have by the multiplication rule \( (1 – x_i – x_j)(1 – x_p – x_q) \geq 0 \) and together with equation (2.1.2) this implies that \( 1 – x_i – x_j – x_p – x_q \geq 0 \) for arbitrary distinct indices \( i, j, p, q \). We can now proceed by induction on the number of rounds. The hypothesis being that after \( k \) rounds one can conclude using \( LS \) all inequalities of the form \( 1 – x_{p_1}… – x_{p_{2k+1}} \geq 0 \) for arbitrary distinct indices \( p_1,…,p_{2k+1} \). Using the \( LS \) multiplication rule as in the case \( k = 1 \) above and equation (2.1.2) one obtains for distinct \( p_1, p_2, ..., p_{2k+2} \)

\[
1 – x_{p_1}… – x_{p_{2k+2}} = (1 – x_{p_1}… – x_{p_{2k+1}})(1 – x_{p_{2k+1}}… – x_{p_{2k+2}}) \geq 0.
\]

Therefore the statement follows when \( n \) is a power of 2. If \( n \) is not a power of 2 we can add dummy variables \( y_i = 0 \) such that \( n \) becomes a power of 2 and the proof carries through. \( \Box \)

**Remark 2.1.3.** The \( LS^+ \) rank of \( SPHP^n_1 \) is 1, i.e. we can derive the integer polytope after one round of \( LS^+ \).

**Proof.** We need to derive the inequality \( 1 – \sum x_i \geq 0 \) using \( LS^+ \) rules. Observe that

\[
f := (\sum x_i - 1)^2 + \sum_{i \neq j} (1 – x_i – x_j)x_i = 1 – \sum x_i
\]

since \( x_i^2 = x_i, \forall i \). Hence the remark follows. \( \Box \)

**Theorem 2.1.4.** After \( k \) rounds of \( SA \) the system of inequalities \( SPHP^n_1 \) has an integrality gap of \( \frac{n}{k+2} \) for the objective function \( f := \max\{x_1 + … + x_n\} \).

**Observation 2.1.5.** Let \( x_i^2 = x_i, i \in [n] \), \( P_i := 1 – \sum_{j \in I_i} x_j \) for \( I_i \subseteq [n] \) and \( I = \bigcup_{i=1}^{k} I_i \). Then

\[
P_I := P_{I_1} P_{I_2}…P_{I_k} = 1 – \sum_{i \in I} x_i + H \quad (2.1.3)
\]
where \( H \) (if non-zero) is a multinomial expression with degree \( \geq 2 \). Furthermore,

\[
\text{coeff}_{\{i\}}(x_i P_I) > 0 \iff i \notin I. \tag{2.1.4}
\]

**Proof.** The proof of equation (2.1.3) follows by induction on \( k \), and that of equation (2.1.4) from elementary observation. \( \Box \)

**Proof of Theorem 2.1.4.** Let \( y_I \) denote the coordinate of the lifted vector corresponding to a subset of the edges \( \Pi_{i \in I} x_i \), after \( k \) lifts in SA. Let \( y_I = \frac{1}{k+2} I \) for \( |I| = 1 \) and \( y_I = 0 \) otherwise. Observe that \( y \) lies in the lifted polytope for \( \text{SPHP}^m \). This is easy to check since any \( k \)-lifted inequality obtained from \( 1 - x_r - x_s \geq 0 \) is of the form

\[
(1 - x_r - x_s) \Pi_{i \in S \subseteq I} x_i \Pi_{j \in I \setminus S} (1 - x_j) \geq 0
\]

for \( I \subseteq [n], |I| \leq k \). On expanding and placing the values for \( y \) we get one of the cases:

1. \(|S| \geq 2\), in which case all terms in the lifted linear inequality correspond to coordinates \( y_I \) with \(|I| \geq 2\) hence we get \( 0 \geq 0 \).

2. \(|S| = 1\), then depending on whether \( r, s \in S \) (or not) eqns (2.1.3) and (2.1.4) reduce the evaluation to \( 0 \geq 0 \) and \( \frac{1}{k+2} \geq 0 \) respectively.

3. Otherwise, we get by equation (2.1.3) and \( y_I = 0 \) for \(|I| \geq 2\), the valuation:

\[
1 - \sum_{i \in I \cup \{r, s\}} y_{\{i\}} \geq 1 - \frac{|I| + 2}{k + 2} \geq 0.
\]

Note that for the same reasons as above \( \bar{y} \) satisfies \( k \)-lifted inequalities obtained from \( x_i \geq 0 \) and \( 1 - x_i \geq 0 \). Since all \( k \)-lifted SA inequalities for \( \text{SPHP}^m \) are positive linear combinations of inequalities lifted from the above three types of inequalities the point \((\frac{1}{k+2}, \ldots, \frac{1}{k+2})\) lies in the projected polytope obtained after \( k \) rounds of SA and the required integrality gap is \( \frac{n}{k+2} \).

**Definition 2.1.6.** [9] Define \( \text{LPHP}^m \) polytope as a Linear encoding of the \( \text{PHP}^m \) principle in definition 2.0.11 by the set of linear inequalities

\[
Q_i := \sum_{j \in [n]} x_{ij} - 1 \geq 0, \ (\forall i \in [m]). \tag{2.1.5}
\]

\[
Q_{jk,i} := 1 - x_{ji} - x_{ki} \geq 0, \ (\forall j \neq k, j, k \in [m], i \in [n]) \tag{2.1.6}
\]

The \( L \) stands for linear.
If \( x_{ij} \in \{0, 1\} \) then \( \text{LPHP}^m_n \) states that there exists a multivalued everywhere defined injective map from \( U \) to \( V \) in \( K_{m,n} \). Therefore the integer polytope for \( \text{LPHP}^{n+1}_n \) is empty. For brevity \( x_{(i,j)} \) is denoted by \( x_{ij} \) whenever there is no scope of confusion.

**Theorem 2.1.7.** The \( \text{LS}_* \) rank of \( \text{LPHP}^{n+1}_n \) is at most \( \lceil \log_2 n \rceil - 1 \).

**Proof.** Again as in the proof of theorem 2.1.2 after \( \lceil \log_2 n \rceil - 1 \) rounds of \( \text{LS}_* \) we can conclude the following for any \( j \in V \):

\[
1 - \sum_{i=1}^{n+1} x_{ij} \geq 0.
\]

Therefore by summing over \( j \in V \) we obtain

\[
n - \sum_{i,j} x_{ij} \geq 0 \quad (2.1.7)
\]

However, for every \( i \in U \) we have \( 1 - \sum_{j \in \Gamma(i)} x_{ij} \leq 0 \). Therefore summing over all \( i \in U \) gives

\[
n + 1 - \sum_{i,j} x_{ij} \leq 0. \quad (2.1.8)
\]

Equation (2.1.7) and equation (2.1.8) give the required contradiction. \( \square \)

It is known that the \( \text{LPHP}^{n+1}_n \) rank of \( SA \) is \( n - 1 \) \cite{19}, that of \( CP \) is \( \log_2 n \) \cite{9} and that of \( LS_+ \) is constant \cite{9, 27}.

**Definition 2.1.8.** Given a cone \( K \), define \( SA_* \) to be a generalization of the \( SA \) hierarchy with the * operator. For \( k \) rounds of \( SA \) the lifted inequality obtained by multiplication of inequalities in \( \mathbb{Q}^* \)

\[
f(\Pi_{s \in S} x_s \Pi_{t \in [n] \setminus S} (1 - x_t))
\]

is replaced by one obtained upon multiplication with inequalities in \( K^* \). i.e.

\[
f(\Pi_{i \in I \subseteq [m]^k} f_i)
\]

where \( f_i \in K^* \).

Note that like \( SA \), projection (if necessary) is applied only at the end of \( k \) rounds of \( SA_* \).

**Observation 2.1.9.** Given a polytope \( K \in \mathbb{R}^n \), the projection (to \( \mathbb{R}^n \)) of the polytope obtained after \( 2^k - 1 \) rounds of \( SA_* \) is a subset of the polytope obtained after \( k \) rounds of \( LS_* \).
Proof. The proof is by induction on \( k \). The initial cases \( k = 0, 1 \) are clear. Assume that any \( LS_\ast \) inequality derived within \( k-1 \) rounds can be expressed as a positive linear combination of projected \( SA_\ast \) inequalities obtained after \( 2^{k-1} - 1 \) rounds. Since all lifted inequalities in round \( k \) of \( LS_\ast \) are generated by multiplying at most two inequalities obtained after round \( k-1 \), induction hypothesis and the definition of \( SA_\ast \) (2.1.8) implies that the resulting lifted inequality can be generated by some positive linear combination of inequalities obtained after \( 2^k - 1 \) rounds of \( SA_\ast \). Hence the proof follows.

Theorem 2.1.10. The \( LS_\ast \) rank of \( SPHP^n \) is at least \( \lfloor \log_2 n \rfloor - 1 \).

Proof. By Observation 2.1.9 it suffices to prove the same result for \( 2^{k-1} - 1 \) rounds of \( SA_\ast \), where \( k < \lfloor \log_2 n \rfloor - 1 \). So it suffices to find a point \( \bar{y} \in \mathbb{R}^{\lfloor n/2 \rfloor} \) in the \( 2^k - 1 \) lifted \( SA_\ast \) polytope for \( SPHP \) such that \( \sum_{i \in \lfloor n \rfloor} y_i > 1 \). Let \( y_i = \frac{1}{2^k+1} \), \( y_I = 0 \) (\(|I| > 1\)).

(2.1.10)

Also \( P_J := 1 - \sum_{i \in J} x_i, \ J \subseteq \lfloor n \rfloor \). Note that \( Q_{ij} \) from equation (2.1.9) is just \( P_{\{i,j\}} \). By definition of \( SA_\ast \) observe that any \( 2^k - 1 \) lifted inequality is obtained from a positive linear combination of the form

\[
\Pi_{J \in I \subseteq \lfloor n \rfloor, |J| \leq 2} \Pi_{J \in S \subseteq \lfloor n \rfloor} x_J
\]

(2.1.11)

such that \(|I| + |S| \leq 2^k \) (\( I \) can be a multiset). Like Theorem 2.1.4 we have 3 cases depending on the cardinality of \( S \):

1. \(|S| \geq 2\), in which case all terms in the lifted linear inequality correspond to coordinates \( y_I \) with \(|I| \geq 2\) hence we get \( 0 \geq 0 \).

2. \(|S| = 1\), then depending on whether \(|I \cap S| > 0\) (or not) eqns (2.1.3) and (2.1.4) reduce the evaluation to \( 0 \geq 0 \) and \( 1 - \frac{1}{2^k+1} \geq 0 \) respectively.

3. Otherwise, we get by equation (2.1.3) and \( y_I = 0 \) for \(|I| \geq 2\), the valuation:

\[
1 - \sum_{i \in \{ \cup_{J \in I} J \}} y_i \geq 1 - \frac{2|I|}{2^{k+1}} \geq 0.
\]

Each of the lifted inequalities is thus satisfied by \( \bar{y} \). Hence \( (y_{\{i\}}) \in \mathbb{R}^n \) belongs to the projected polytope of \( SA_\ast \) and therefore the statement follows.

Another proof of theorem 2.1.10 follows from the following lemma. As opposed to exhibiting a point it completely characterizes the lifted polytopes for \( SPHP \) in \( LS_\ast \). The proof is deferred to the appendix.
Lemma 2.1.11. After $k \geq 0$ rounds of LS$_*$ for SPHP$^n_1$ all linear inequalities generated are some positive linear combination of

$$1 - \sum_{i \in [n]} x_i \geq 0, \ 0 \leq \gamma \leq 2^k \tag{2.1.12}$$

$$x_i \geq 0. \tag{2.1.13}$$

Theorem 2.1.17 proves a linear rank lower bound in SA$_*$ for LPHP. The strategy of the proof will again be to show that a certain lifted polytope is non-empty i.e. it contains at least one point. However, since the construction of the point is a little intricate we describe it in function notation. The evaluation is essentially a function $x_I \to \mathbb{R}$ and describes the point that shows the lifted LPHP polytope to be non-empty after $\Omega(n)$ lifts of SA$_*$. It is the same one used in [19] but is restated below to ensure consistent notation.

Definition 2.1.12 ([19]). A partial bijection in $P_{n-1}^n$ is a bijection $[2..n+1] \setminus \{i\} \to [2..n]$ for $i \in [2..n+1]$. $P_{n-1}^n$ denotes the set of partial bijections from $[2..n+1] \to [2..n]$.

Definition 2.1.13 ([19]). Given $I \subseteq \{(p,q) | p \in [2..n+1], q \in [2..n]\}$, $I$ is self-inconsistent if $(i,j), (i,k) \in I$ and $j \neq k$, or $(j,i), (k,i) \in I$ and $j \neq k$.

If $I$ is not self-inconsistent then $I$ is self-consistent. Intuitively a self-consistent set $I$ naturally corresponds to a bijection of two sets of size $|I|$.

Definition 2.1.14 ([19]). Given $I \subseteq \{(p,q) | p \in [2..n+1], q \in [2..n]\}$, $I$ is inconsistent with $\pi \in P_{n-1}^n$ if either $I$ is self-inconsistent or

- $(i,j) \in I$ and $\pi(k) = j$ for $i \neq k$
- $(i,j) \in I$ and $\pi(i) = l$ for $j \neq l$.

If $I$ is not inconsistent with $\pi$ then $I$ and $\pi$ are consistent with each other. Intuitively a restriction of $\pi$ would correspond to the bijection represented by $I$. For brevity let $(*)$ denote the wildcard character (for example $(r,*) \in I$ stands for the set $\{(r,a) \in I | a \in [2..n]\}$ being non-empty) and let $N = n^2 + n$ i.e. the number of variables in LPHP$^n_{n+1}$. Now we define the evaluation function i.e. the lifted point.

Definition 2.1.15 ([19]). An evaluation $V : x_I \to \mathbb{R}$ is a function defined on all lifted variables obtained from monomials of degree at most $n-1$ and linearly extended to the lifted inequalities. For $I \subseteq \{(p,q) | p \in [2..n+1], q \in [2..n]\}, |I| \leq n - 1$ define $V(x_I)$ as the fraction of all $n!$ partial bijections $P_{n-1}^n$ consistent with $I$. 

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If $(1, i) \in I$ (resp. $(i, 1) \in I$) then $V(x_I) := V(x_{(r,i):I})$ (resp. $V(x_I) := V(x_{(i,r):I})$), where $(r, i) : I$ (resp. $(i, r) : I$) denotes $(r, i)$ (resp. $(i, r)$) substituted for all instances of the form $(1, i)$ (resp. $(i, 1)$) in $I$ and $(r, *) \notin I$ (resp. $(*, r) \notin I$). Note that such an $r$ exists since $|I| \leq n - 1$.

Observe (by symmetry arguments) that $V$ is well defined for all monomials in $S_N(\mathbb{R})$ of degree at most $n - 1$. If one substitutes the exact definition for $LPH P_{n+1}$ from that of [19], interchanges $1 \leftrightarrow n + 1$, and adds the condition that $(*, 1) \notin I$ for $I$ to be self-consistent then one obtains exactly the evaluation function from [19]. The following lemma is implicit in the proof of proposition 11 in [19].

**Lemma 2.1.16.** Let $Q_i$ be defined as in equation (2.1.5) then $V(x_{I}Q_{i}) = 0$ for any monomial $x_{I}$ and $|I| \leq n - 2$.

**Proof.** First suppose $i \neq 1$ and $(1, *), (*, 1) \notin I$. Observe that $I$ is self-consistent otherwise the statement of the Lemma follows immediately. Let $P'_{i}$ denote the set of $\pi \in P_{n-1}$ consistent with $I$ such that $i$ remains unmatched. By defn 2.1.15

$$
\sum_{j=2}^{n} V(x_{I\cup(i,j)}) + \frac{|P'_{i}|}{n!} = V(x_{I}). \tag{2.1.14}
$$

Equation (2.1.14) is true since either

1. $(\exists a \in [2..n])(i, a) \in I$ then observe that by the definition of evaluation: $(\forall b \neq a)(V(x_{I\cup(i,b)}) = 0)$, $P'_{i} = 0$ (since $\pi' \in P'_{i} \Rightarrow ((\exists l \neq i)\pi'(l) = a)$, and $V(x_{I\cup(i,a)}) = V(x_{I})$. Note that in this case the statement of the Lemma follows so we may assume $(i, *) \notin I$ from now.

2. Otherwise, equation (2.1.14) follows from definition of $V$.

Observe that $|I| \leq n - 2 \Rightarrow ((\exists l \in [2..n])(*, l) \notin I)$ so that $V(x_{I\cup(i,1)}) := V(x_{I\cup(i,l)})$. Note that $I$ self-consistent and $(i, *) \notin I$ implies that $I \cup (i, l)$ is self-consistent. It now suffices to show that there is a bijection between $P'_{i}$ and the set of partial bijections consistent with $I \cup (i, l)$ (denoted by $P_{i,l}$).

To see $|P_{j,l}| \geq |P'_{i}|$, observe that for $\pi' \in P'_{i} \Rightarrow (\exists l' \in [2..n+1])(i' \mapsto l)$ one replaces $i'$ by $i$ to obtain a unique $\pi \in P_{j,l}$. The consistency of $\pi$ with $I \cup (i, l)$ follows from consistency of $\pi'$ with $I$ and the construction of $\pi$.

---

**ii**The negative atoms in [19] together with the equality of negation correspond to the linear extension property in definition 2.1.15. Note we don’t use negative atoms.

**iii**Dantchev et al [19] only prove injection and it suffices for their purposes. It is unlikely that the proof of theorem 2.1.17 would work if only injection were true, in fact the point described by $V$ would not lie in the polytope obtained after 1 round of $LS$. 

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To see $|P_d| \leq |P'_d|$, given $\pi \in P_d \Rightarrow ((\exists i' \in [2..n + 1] \setminus \{i\})(i' \notin \text{Dom}(\pi))$ replace $i$ by $i'$ to obtain a unique $\pi' \in P'_d$. The consistency of $\pi'$ with $I$ follows from consistency of $\pi$ with $I \cup (i, l)$.

Therefore $|P'_d| = V(x_{I \cup (i, l)})$ so the statement is true when $i \neq 1$ and $(1, \ast), (\ast, 1) \notin I$.

Otherwise, if $i = 1$ or $(1, a) \in I$ or $(a, 1) \in I$ then we can reduce this case to the previous case by substituting $i' \in [2..n + 1], j' \in [2..n]$ such that $(i', \ast), (\ast, j') \notin I \cup (i, j)$, in place of 1. After the substitutions note if $I$ is not self-consistent or $(i, \ast) \in I$ then $V(x_{I^j Q_l}) = 0$ immediately follows. Otherwise, if $i = 1$ or $(1, a) \in I$ then substituting 1 with $i'$ reduces this case to when $i \neq 1$ and $(1, \ast) \notin I$ respectively. If $(a, 1) \in I$ then observe that $\sum_{j=1}^{n} V(x_{I \cup (i, j)}) = V(x_{(a, j'; I \cup (i, j'))}) + \sum_{j=1, j \neq j'}^{n} V(x_{(a, j'; I \cup (i, j))})$. By swapping $j'$ and $j''$ in $(a, j'') : I \cup (i, j')$ we now need to prove the equation $V(x_{(a, j''); I^j Q_l}) = 0$. Therefore if $i = 1$ or $(1, \ast) \in I$ or $(\ast, 1) \in I$ then we are able to reduce all such cases to the previous one. Hence the proof follows. \hfill \Box

**Theorem 2.1.17.** Deriving the empty integer polytope for $LPHP_n^{n+1}$ requires rank at least $\lceil \frac{n}{2} \rceil - 2$ in $SA_*$. 

*Proof.* Let $Q_i$ denote the form in equation (2.1.5) and $Q_{jk,l}$ denote the form in equation (2.1.6). A rank $k_*$ $SA_*$ form is derived from an expression of the form

$$F := \Pi_{i \in S_1} Q_i \Pi_{(jk,l) \in S_2} Q_{jk,l} \Pi_{p \in S_3} (1 - x_p) \Pi_{q \in S_4} x_q$$

where the meaning of the sets $S_i$ is intuitive, $\sum_{i=1}^{4} |S_i| \leq k_* + 1$, and $F \in S_N(\mathbb{R})$. Observe that

$$Q_{jk,l} = 1 - x_{jl} - x_{kl} = (1 - x_{jl})(1 - x_{kl}) - x_{jl}x_{kl}.$$ 

Any evaluation of $F$ which is defined by a linear combination of its value on monomials will be invariant under the above rewrite. Therefore we can rewrite $F$ as a linear combination of forms with no $Q_{jk,l}$ without changing $V(F)$, furthermore the degree of any form in the combination is still at most $\sum_{i=1}^{4} |S_i| + |S_2| \leq 2k_* + 2$. Therefore any positive linear combination of forms $F$ after the above rewrite resembles:

$$F' := \sum_i c^+ |\sum_i (Q_i g_i) + \sum_{j \neq k,l} (x_{jl}x_{kl} h_{jk,l}) + \sum_{p,q} \Pi_{p \in S}(1 - x_p) \Pi_{q \in T} x_q|$$

such that $(\forall g_i \in S_N(\mathbb{R}))(\deg(g_i) \leq 2k_*), (\forall h_{jk,l} \in S_N(\mathbb{R}))(\deg(h_{jk,l}) \leq 2k_*), |S| + |T| \leq 2k_* + 2$ and $c^+ \in \mathbb{R}^+$. Now suppose one could refute $LPHP$ by a rank $k_*$ ($k_* = \lceil \frac{n}{2} \rceil - 2$) $SA_*$ proof i.e. one could derive $-1$ by some positive linear combination of lifted inequalities derived from forms of type $F'$. At this point it suffices to prove that $V$ (from definition 2.1.15) has $V(F') \geq 0$, since it would give an immediate contradiction.

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By definition 2.1.15 $V$ is linear and $(\forall j, k, l)(V(x_{jl}x_{kl}h_{jk,l}) = 0)$. Furthermore Lemma 2.1.16 implies that $(\forall i)(V(Q_{ig}) = 0)$ hence

$$V(F') = \sum c^+ V(\sum_{p,q} \Pi_{p \in S}(1 - x_p)\Pi_{q \in T} x_q). \tag{2.1.16}$$

However equation (2.1.16) can be derived by $2k_s + 2$ rounds of SA alone so the fact that it is $\geq 0$ follows from the result of Dantchev et al. [19] ([19] proves a lower bound of $n - 1$ on the rank of $LPHP^*_n$ in SA). This can (also) be seen directly by observing that $V(x_l\Pi_k(1 - x_{i_k,j_k}))$ is the fraction of partial bijections consistent with $I$ and where no $i_k$ is matched to $j_k$. Therefore the statement follows.

**Corollary 2.1.18.** The $LS_s$ rank of $LPHP^*_n$ is at least $\lfloor \log_2(n-3) \rfloor - 1$.

**Proof.** The result follows from Observation 2.1.9 and theorem 2.1.17. \qed

Note that the bound above extends to the weaker functional-$PHP$ [42] but not to onto-$PHP$ [42] since Remark 2.1.3 implies that the later has $LS_s$ rank 1.

### 2.2 MAX-CUT and related bounds

In this section the aim is to extend the rank lower bounds for MAX-CUT and associated problems proved for $SA$ hierarchy by Charikar et al [12] to rank lower bounds in the $SA^*$ hierarchy and therefore to $LS^*$ hierarchy via Observation 2.1.9. Throughout this section $G = (V(G), E(G))$ will represent a graph on $n$ vertices.

**Definition 2.2.1 ([21]).** Given graph $G$ and distributions (i.e. discrete probability measure) $\mu_T$ on cuts (i.e. subsets) of $T \subseteq V(G)$ for every $T$ such that $|T| \leq k$, the distributions $\mu_T$ are $k$-locally consistent if for any $A, Q, T$, $A \subseteq Q \subseteq T$ implies $\mu_T(\{B|B \subseteq T, B \cap Q = A\}) = \mu_Q(A)$.

Charikar et al. [12] deduce that given a linear program it is sufficient to prove the existence of ‘locally consistent’ probability distributions on subsets of size $k$ to show $\Omega(k)$ rank lower bound in the $SA$ hierarchy for the original linear program.

**Lemma 2.2.2 ([21, 12]).** Given $k$-locally consistent probability distribution of cuts on graph $G$, the vector $x_{ij} = \mu_{\{i,j\}}(\{i\}, \{j\})$ (i.e. equivalently interpretable as the probability that $i, j$ lie on different sides of the cut) lies in the $SA$ cut polytope obtained after $k^2 - \frac{k}{2}$ rounds.

Furthermore Charikar et al. [12] show that the existence of certain discrete metric spaces (in particular metrics which are isometrically embeddable
in $\ell_2$ but have high distortion when embedded in $\ell_1$) leads to the construction of $\Omega(n^\gamma)$-locally consistent distributions such that the vector $\bar{x} \in \mathbb{R}^{[n]^2}$, where $x_{ij} = \mu_{\{i,j\}}(\{\{i\}, \{j\}\})$, is not a convex combination of cut metrics.

**Lemma 2.2.3** ([11]). For every $n$ and $k < n$ there exists a bounded degree expander $G$ on $n$ vertices such that the metric space equipped with the distance

$$\rho(u, v) = \sqrt{1 - (1 - \nu)d(u, v) + \nu}$$

where $\nu = \Theta(\frac{\log k + \log \log n}{\log n})$ satisfies

1. Every embedding of $G$ into $\ell_1$ requires $\Omega(\frac{1}{\sqrt{\nu}})$ distortion.
2. Every subset of $V$ of size at most $k$ embeds isometrically into $\ell_2$.

Lemma 2.2.3 (and some versions of it) were used in [12] to construct a locally consistent distribution for MAX-CUT, Sparsest cut, Vertex cover and also Unique games. Hence [12] shows rank lower bounds for all the above mentioned problems in the $SA$ hierarchy (in fact they even show strong integrality gaps for the same number of rounds).

We will only work with the MAX-CUT linear program in this section. However, the lower bounds extend to related problems (like Vertex Cover, Sparsest Cut etc) as in [12].

**Definition 2.2.4.** Let $x_{ij} \in [0, 1]$ denote variables corresponding to vertices $i, j$ ($i \neq j$) in an undirected graph $G$.

$$x_{ij} \geq 0, 1 - x_{ij} \geq 0, x_{ij} = x_{ji}, \quad (2.2.1)$$

$$x_{ij} + x_{jk} - x_{ik} \geq 0 \quad \forall i, j, k \in V(G), \quad (2.2.2)$$

$$2 - x_{ij} - x_{jk} - x_{ik} \geq 0 \quad \forall i, j, k \in V(G). \quad (2.2.3)$$

The above polytope is referred to as the cut polytope.

In order to generalize the above mentioned two step approach to the $SA_*$ hierarchy one just needs to generalize Lemma 2.2.2. The proof closely mimics the proof in the appendix of [12] (differing only in two lines) but is restated for purposes of later reference.

**Lemma 2.2.5.** Given $k$-locally consistent distribution of cuts on graph $G$, the vector $x_{ij} = \mu_{\{i,j\}}(\{\{i\}, \{j\}\})$ (i.e. equivalently probability that $i, j$ lie on different sides of a cut) lies in $SA_*$ cut polytope obtained after $\frac{k}{\gamma} - 1$ rounds.

**Proof.** Consider a non-empty set $I = \{(i_1, j_1), ..., (i_p, j_p)\}$ of size at most $\frac{k}{2}$. Let $Q_I$ denote the set $\{i_1, j_1, ..., i_p, j_p\}$ of size at most $k$. Let $1_{ij}(X)$ denote an indicator variable that is 1 if the cut $X$ on $Q_I$ separates vertices $i$ and $j$
for $i,j \in Q_I$ and $0$ otherwise. The coordinates of the $SA_*$ solution vector $ar{y} \in \mathbb{R}^{n+1}$ are defined as follows:

$$y_I = E_{\mu_{Q_I}}(\Pi_{(i,j) \in I} 1_{ij}).$$

In other words $y_I$ is the probability that all pairs of vertices in $I$ are separated by a cut chosen according to $\mu_{Q_I}$.

Let $R(\{y_{ab}, y_{cd}, y_{ef}\})$ denote an initial constraint in the MAX-CUT linear program (for eg. 2.2.2 or 2.2.3) evaluated at $\{y_{ab}, y_{cd}, y_{ef}\}$. For brevity we will use $R_i(y)$ to denote some initial constraint $R_i$ evaluated at some triple of coordinates $\{y_{ab}, y_{cd}, y_{ef}\}$ chosen from $y$.

Consider an inequality obtained after $r$ lifts of $SA_*$:

$$\Pi_{p \in \mathcal{I}} R_p. \quad (2.2.5)$$

Let $Q_I = \bigcup_{i \in R_p, p \in \mathcal{I}} i$ denote the set of vertices present as indices in the above lifted inequality. Observe that the lifted variables in the lifted inequality above will have the form $x_I$ where $|I| \leq r + 1$ and $Q_I \subseteq Q_I$. If $|Q_I| \leq k$ then there exists a $k$ set $Q_S$ such that $Q_I \subseteq Q_S$. This condition implies:

$$3(r + 1) \leq k \quad (2.2.6)$$

Next, using linearity of expectation and $k$-local consistency one can simplify and evaluate the lifted inequality at $\bar{y}$ as follows.

$$\Pi_{p \in \mathcal{I}} R_j(u) = E_{\mu_S}(\Pi_{p \in \mathcal{I}} R_j(\{1^S_{ab}, 1^S_{cd}, 1^S_{ef}\})) \quad (2.2.7)$$

Since $\Pi_{p \in \mathcal{I}} R_p(X) \geq 0$ for any given cut $X$ on $Q_S$ the expectation on RHS above is non-negative. Therefore $\bar{y}$ lies in the $SA_*$ polytope for MAX-CUT obtained after $\frac{k}{3} - 1$ rounds.

Therefore we get the following result.

**Theorem 2.2.6.** The $LS_*$ rank of MAX-CUT is $\Omega(\log_2 n)$.

The two step approach of Charikar et al [12] (esp. Lemma 2.2.2) is quite general and can be used to prove integrality gaps for linear programming problems other than MAX-CUT. Unlike $SA$, a generalization of Lemma 2.2.5 to $SA_*$ i.e., existence of $k$-locally consistent distributions implies $\Omega(k)$ rank lower bound in $SA_*$ hierarchy, is not possible for all linear programming relaxations. However, a $\frac{k}{c} - t$ rank lower bound in the $SA_*$ hierarchy follows by modifying the corresponding $\frac{k}{c} - t$ $SA$ rank lower bound, where the constant $c$ depends on the problem at hand ($c$ is 2 for MAX-CUT) and $t$ is the maximum number of variables in any inequality of the initial linear program. In each problem the only modification required would be to replace the factor 3 in equation 2.2.6 by $t$ in the corresponding $SA$ proof. Note that $t = \sqrt{n}$ for $L PHP_n^{n+1}$ so the $SA_*$ rank lower bound obtained by immediately using the above approach would not be tight for $L PHP$. We end this section with the following conjecture (which seems quite difficult to the author).

**Conjecture 2.2.7.** The $LS_*$ rank of MAX-CUT is $\omega(\log n)$.  

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2.3 Further comparisons of $LS_\ast$ rank

In this section the aim is to show that the $LS_\ast$ rank and $LS_+\ast$ rank (or even $SA$ rank) are incomparable.

**Definition 2.3.1.** Let $K_m^n$ denote the polytope in $\mathbb{R}^n$ defined by the following two constraints

$$g_m := \sum_{i=1}^{n} x_i - m \geq 0, \ m \in (0,1); \ f := 1 - \sum_{i=1}^{n} x_i \geq 0. \quad (2.3.1)$$

Note that $(K_m^n)_I := \{ x \in [0,1]^n : \sum x_i = 1 \}$ denotes the required integer polytope.

**Theorem 2.3.2.** $\forall m \in (0,1)$ the $LS_\ast$ rank of $K_m$ is 1.

**Proof.** Observe that using the multiplication and summation rules of $LS_\ast$ give

$$(\sum_{i=1}^{n} x_i - m)(1 - \sum_{i=1}^{n} x_i) + \sum_{i=1}^{n} (x_i^2 - x_i) + \sum_{i\neq j} x_i x_j = m \sum_{i=1}^{n} x_i - m$$

which immediately gives $(K_m^n)_I$.

Next, observe that Cheung [13](also Cook and Dash [18]) prove that the $LS_\ast$ rank of the polytope $g_m$ (for $m \in (0,1)$) is $n$. The proof simply checks that there exists a point $y \in \mathbb{R}^{(n+1)^2}$ in the lifted polytope $N_{n-1}^n(g_m)$ such that $\sum_{i\in[n]} y_{(i)} < 1$. One now merely has to check that the same $(y_{(i)}) \in N_{n-1}^n(K_m)$. For convenience the entire argument (with simply $g_m$ replaced by $K_m$) is restated in the proof of Theorem 2.3.3 (deferred to the appendix).

**Theorem 2.3.3.** $\forall m \in (0,1)$ the $LS_\ast$ rank of $K_m$ is $n$.

The following corollary is implied by the proof of theorem 2.3.3 given in the appendix.

**Corollary 2.3.4.** For all $m \in (0,1)$ and the objective function $\min\{\sum x_i\}$ over $K_m$, the integrality gap of $K_m$ after $k$ rounds of $LS_+\ast$ is $\frac{m}{1-(1-m)^2/n}$. 

In general the following remark based on theorem 6.1 in Goemans et al. [25] is true. Given cones $K_1, K_2$ let $T(K_1, K_2) := \{uv^T + vu^T : u \in K_1^*, v \in K_2^*\}$ and $D := \{Y \in S^{d+1} : \text{diag}(Y) = Ye_0\}$. As usual $M_x$ denotes the lifted cone corresponding to $LS_x$. The proof by replacing $T(K,K)$ for $T(K)$ in theorem 6.1 of [25].

**Remark 2.3.5.** Let $K \subseteq Q$ be a convex cone. Then

$$T(K,K) + D^* \supseteq S^{d+1}_{+} \iff M_x(K) \subseteq M_+(K) \iff M_x(K) = M_+\ast(K)$$
where $S^{d+1}_+$ is the interior of the cone $S^{d+1}$ (positive semidefinite matrices) and

$$N_+(K) \subseteq N_+(K) \Rightarrow \forall y \in \mathbb{R}^{d+1} \mid \text{Diag}(y) \in T(K, Q) + D^\perp + S^{d+1}_+ \Rightarrow \text{Diag}(y) \in T(K, K) + D^\perp).$$

Cheung [13] shows that the polytope $g_m$ contains the point

$$y(i) = \frac{d}{dn + 1 - d}, \quad |I| \geq 2 \Rightarrow y_I = 0 \quad (2.3.2)$$

after $n - 1$ lifts of SA.

**Observation 2.3.6.** The point $y$ in (2.3.2) belongs to $K_m$ even after $n - 1$ lifts of SA.

**Proof.** Since one merely needs to check that the point in (2.3.2) also satisfies the lifted SA inequalities for $f$ in (2.3.1) the proof follows from Observation 2.1.5 and is similar to that of theorem 2.1.10.

It would be interesting to compare $LS_*$ relaxations with Lasserre relaxations in the same vein as above.

**Conjecture 2.3.7.** There exists a system of inequalities which requires constant number of lifts in the $LS_*$ hierarchy and $\Omega(n)$ lifts in the Lasserre hierarchy to converge to the integer polytope.
Chapter 3

Open Problems

Besides the already mentioned conjectures two other open problems are mentioned below.

There are no known super-logarithmic bounds for $LS_*$ for “natural” problems. In case of other $LS$ hierarchies we can verify (via an inductive argument) that a given point lies within the polytope obtained after $r$ rounds of relaxation. This is relatively easy since one only needs to verify it for a known set of vectors - the cone $Q^*_n$. In case of $SA$ (and even Lasserre) hierarchies the structure of the linear program or semidefinite program after $r$ rounds of relaxation is known so again it is possible to prove a lower bound on the rank by explicitly verifying whether a given point lies in the relaxed polytope or not. However, after $\log_2 n$ rounds of $LS_*$ neither of the two approaches above seems to be work and it seems that one would need some characterization of the intermediate projected cones akin to Lemma 2.1.11. Perhaps such a characterization can be more easily obtained for simpler sets of inequalities (as opposed to NPC problems) like Symmetric Knapsack. So we repeat the question from Grigoriev et al. [26] but with a more pessimistic tone.

**Conjecture 3.0.8.** Any refutation of Symmetric Knapsack ($SK_n$) requires $\omega(\log n)$ rank in $LS_*$ hierarchy.

Thus far we have not investigated results which have the conjunction of semidefinite operator and $*$ operator. In fact some lower bounds (which are believed to be true) for problems like vertex cover, unique games [14] are still not known in $LS_+$ hierarchy. It would be interesting to improve the already existing lower bounds in the presence of semidefiniteness constraint. It would also be interesting to extend the known results [39] for semidefinite operator with the $*$ operator (the method of Charikar et al. is not helpful in presence of semidefiniteness) even for $\log n$ rounds of relaxation.
Bibliography


Appendices
An alternative short proof of theorem 2.1.4 is given below.

*Alternative Proof of Theorem 2.1.4.* First one defines a set of “local” probability distributions on subsets of edges of cardinality at most $k+1$ as follows. For $S \subseteq [n]^{k+1}$ with probability $\frac{|S|}{k+2}$ set $x_u = 1$ for $u \in S$ chosen uniformly at random, and set $x_v = 0, \forall v \in S, v \neq u$. In the remaining case with probability $1 - \frac{|S|}{k+2}$ set $x_v = 0, \forall v \in S$.

Therefore $\forall u \ P(x_u = 1) = \frac{1}{k+2}$ w.r.t. any of the distributions above. Also the distributions are locally consistent as $P_S(x_I = 1) = P_T(x_I = 1)$ for $I \subseteq S \subseteq T$. Therefore by the result of Charikar et al. [12] one gets the required integrality gap above.

*Proof of Lemma 2.1.11.* The proof proceeds by induction on $k$ - the number of rounds of $L_S$. The base case $k = 0$ clearly holds. Suppose the statement is true for $k$ rounds of $L_S$. Observe that any linear form in $S_n(\mathbb{R})$ generated in the $k+1$th round will be a positive linear combination of 3 types of terms:

1. $(1 - \sum_{i \in I} x_i)(1 - \sum_{i \in J} x_i)$ for $|I|, |J| \leq k$.
2. $(1 - \sum_{i \in I} x_i)x_j$ with $0 \leq |I| \leq k$
3. and $x_i x_j$ for $i \neq j$.

Any expression of type 1 can be rewritten as

$$1 - \sum_{i \in I \cup J} x_i + \sum_{i \in I, j \in J, i \neq j} x_i x_j.$$ 

Observe that if $j \not\in I$ then any term of type 2 can be rewritten as

$$(1 - \sum_{i \in I} x_i)x_j = x_j - \sum_{i \in I} x_i x_j$$

otherwise it can be written as

$$(1 - \sum_{i \in I} x_i)x_j = -\sum_{i \neq j} x_i x_j.$$ 

Therefore the linear form will be a linear combination of

1. $(1 - \sum_{i \in K} x_i)$ for $|K| \leq 2k$ with positive multiples only
2. $x_i$ with positive multiples only
3. and $x_i x_j$ for $i \neq j$ with positive and negative scalar multiples.

Since the final form is linear in $S_n(\mathbb{R})$, each of the quadratic terms of the form $x_i x_j$, $(i \neq j)$ must have the sum of their coefficients evaluate to 0. Hence the proof follows.  \[\square\]
The following definitions are required for the proof of theorem 2.3.3. Let $e_i$ denote the $i$th standard unit vector (where the dimension will be clear from the context) and let $e$ denote the all 1s vector. For $a \in \mathbb{R}^{n+1}$ define $\bar{a} \in \mathbb{R}^n$ as $a = (1, a)$. Let $F_i^0$ denote the face of $Q$ with $i$th coordinate set to 0.

**Definition .0.9** ([18]). Define embedding $\text{emb}_I : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$ such that if $y = \text{emb}_I(x)$ then $y_{ij} = x_j$ for $I := \{ i_j \in [n+k] | j \in [n] \}$, and $y_{ij} \in \{0, 1\}$ for $i \notin I$.

For a face $F$ of $Q_n$ let $\text{emb}_F$ denote the embedding where $Q_{\dim(F)} \leftrightarrow F$ (i.e. $\text{emb}_F$ is short for $\text{emb}_{I}$, $I = [n] \setminus \{ i \}$ where $i$ is the coordinate fixed to $\{0, 1\}$ in $F$). Lemma 2.1 in Cook and Dash [18] is restated below.

**Lemma .0.10** ([18]). Given polytope $P \subseteq Q$ and embedding $\text{emb} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $N_+(\text{emb}(P)) = \text{emb}(N_+(P))$.

**Proof of Theorem 2.3.3.** It suffices to show $y^{k,n}e_0 \in N_+^k(K^n_m)$ for $k < n$ and some $y^{k,n} \in \mathbb{R}^{(n+1) \times (n+1)}$ defined below. Let

$$y^{k,n}_{0,0} = 1, y^{k,n}_{i,i} = y^{k,n}_{0,0} = y^{k,n}_{i,i} = \frac{m}{n - (1 - m)k}$$

and $y^{k,n}$ is 0 elsewhere. Hence $y^{k,n}$ is a symmetric, positive semidefinite (diagonally dominant) matrix for all $k < n$. Note that $\sum_{i \in [n]} y^{k,n}_{i,i} < 1$ as required.

The proof proceeds by induction on $k, n$. In the base case $n = 1$ and $k = 0$ the hypothesis holds. Suppose the induction hypothesis (i.e. $y^{k,n}e_0 \in N_+^k(K^n_m)$ for $k < n$) holds for $K^n_m$ with $n \geq 2$ and $k < n$.

For brevity let $y^{k,n}$ defined above be denoted by $y$. Observe that $\bar{y}e_i$ is a positive multiple of $e_i \in (K^n_m)$, therefore $\bar{y}e_i \in N_+^{k-1}(K^n_m)$ for $k < n$.

Let $z_i = y(e_0 - e_i)$. Then

$$z_i = \frac{m}{n - (1 - m)k - m} e_0 + \sum_{j=1,j \neq i}^n \frac{m}{n - (1 - m)k - m} e_j = \bar{z}_i = \sum_{j=1,j \neq i}^n \frac{m}{n - (1 - m)k - m} e_j.$$ 

So $\bar{z}_i \in F_i^0$. Also $\frac{m}{n - (1 - m)k - m} e = \frac{m}{n - (1 - m)(k-1)} e$ hence by induction hypothesis

$$\frac{m}{n - (1 - m)k - m} e \in N_+^{k-1}(K^n_m) \Rightarrow \bar{z}_i \in \text{emb}_{F_i^0}(N_+^{k-1}(K^n_m)) = N_+^{k-1}(K^n_m \cap F_i^0) \subseteq N_+^{k-1}(K^n_m).$$ 

The equality on the the RHS of the implication above follows from Lemma .0.10 and the observation $K^n_m \cap F_i^0 = \text{emb}_{F_i^0}(K^n_m)$. Hence $y \in M_+(N_+^{k-1}(K^n_m))$ for $k < n$ and so $(y_{i,i}) \in N_+^k(K^n_m)$. Hence the proof follows.